

## On the number of Bounding Cycles for Nonlinear Arrangements

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*To Peter Orlik on his sixtieth birthday*

For a real hyperplane arrangement  $A \subset \mathbb{R}^n$ , among the first invariants that were determined for  $A$  were the number of chambers in the complement  $\mathbb{R}^n \setminus A$  by Zaslavsky [Za] and the number of bounded chambers by Crapo [Cr]. In the consideration of certain classes of hypergeometric functions, there also arise arrangements of hypersurfaces which need not be hyperplanes (see e.g. Aomoto [Ao]). In this paper we will obtain a formula for the number of bounded regions (i.e. chambers) in the complement of a nonlinear arrangement of hypersurfaces. For example, for the general position arrangements of quadrics in Figure 1, we see the number of bounded regions in the complement are respectively 1, 5, and 13.

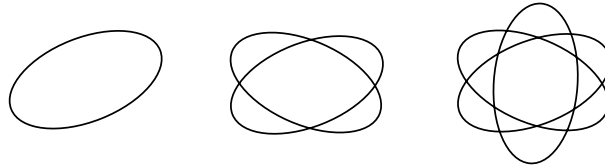


Figure 1

A computation of the number of bounded regions in the complement depends on the degrees of the hypersurfaces as well as the combinatorial structure of the arrangement. Hence, the form such a formula should take is less obvious, even given the answer for hyperplane arrangements. Moreover, in the real case for hypersurfaces of degree  $> 1$  there is the added complication that the number depends upon the specific hypersurfaces (another choice of real quadrics could have fewer real intersections).

In the case of real hyperplane arrangements, the number of bounded regions in the complement represents an intrinsic invariant for the associated complex arrangements. Each bounded region has a bounding

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cycle, and these cycles represent the nontrivial cycles in the associated complex hyperplane arrangement. For arrangements of hypersurfaces of degree  $> 2$ , there is the added complication that the number of bounded regions does not accurately count the number of “bounding cycles” for the complexification. For example, the arrangement in figure 2 of a quadric and elliptic curve has a maximum of 6 bounded regions while by [D1, §6], the number of “bounding cycles” for the complexification is 8. In fact, it is the number of “bounding cycles” for arrangements of smooth complex hypersurfaces  $A \subset \mathbb{C}^n$  which is intrinsic and we shall refer to these cycles as the “bounding cycles”.

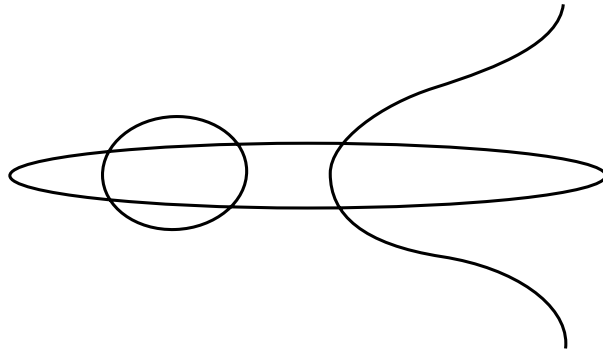


Figure 2

In order to obtain a formula for the number of bounding cycles, we are led to consider more generally a nonlinear arrangement of hypersurfaces  $A$  on a smooth complete intersection  $X \subset \mathbb{C}^n$ , and consider the corresponding number of “relative bounding cycles” for  $(X, A)$ . Moreover, we consider nonlinear arrangements which are the analogues of arbitrary hyperplane arrangements rather than just general position arrangements. Then, in Theorem 1, we shall give a general formula (5.13) for the number of “relative bounding cycles” for a nonlinear arrangement  $A$  of  $q$ -tic hypersurfaces on a smooth complete intersection  $X$  of multidegree  $\mathbf{d}$ , where the nonlinear arrangement is generally based on any central hyperplane arrangement  $\mathcal{A}$ . The formula is valid for general nonlinear arrangements provided that the arrangement and complete intersection are “nondegenerate at infinity” (see §1). It has the form

$$d \cdot \left( \sum_{j=0}^r \alpha_j \cdot q^{r-j} \mu_{r-j}(\mathcal{A}) \right)$$

Here,  $d$  is the product of the multidegrees and the coefficient  $\alpha_j$  involves  $s_j$ , the  $j$ -th complementary function to the elementary symmetric functions, applied to the multidegree  $\mathbf{d}$  and common arrangement degree  $q$ . Also,  $\mu_{r-j}(\mathcal{A})$  are the higher multiplicities of the arrangement  $\mathcal{A}$  [D2, §4]. The higher multiplicities are certain intrinsic geometric invariants of the central arrangement, which are the analogues of those originally introduced by Teissier [Te] for hypersurfaces (see §2).

Several key ideas play crucial roles in obtaining such a formula. In §1 we reduce computation of global invariants to local invariants of an appropriate mapping (§1). Then, the number of “relative bounding cycles” turns out to be a “relative singular Milnor number” which, in turn, is sum of the usual Milnor number for an isolated complete intersection singularity and the “singular Milnor number” for the intersection of the arrangement and the complete intersection. In the special case that such a nonlinear arrangement is based on a free arrangement, formulas were given in [D1] for the special case where  $X = \mathbb{C}^n$  or  $X$  is a smooth complete intersection but the arrangement consists of hyperplanes. We recall these formulas in §5. To generalize these formulas for all central arrangements rather than just free arrangements, we introduce a version of “nonlinear deletion–restriction” (§3). The version we give does not yield analogues of the complete results obtained by Orlik–Terao [OT] for the topology of complements of hyperplane arrangements; however, it suffices for counting the number of bounding cycles. It leads to functional equations (4.1) which such a formula for the number of bounding cycles must satisfy (§4). The form of the solution to these functional equations is obtained in terms of the higher multiplicities of the central arrangement and the multidegree of the complete intersection  $X$ . It is obtained by expressing the formulas valid for free central arrangements in terms of higher multiplicities (§5). The proof that the formula satisfies the functional equations is given in §6. As a consequence, it follows that the formulas which originally were obtained for the special case of nonlinear arrangements based on free arrangements, when reexpressed in terms of higher multiplicities, are seen to hold for all nonlinear arrangements.

In all that follows, we shall use standard notation and terminology for hyperplane arrangements as given in [OT], especially chapters 1 and 2.

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### §1. Nonlinear Arrangements Nondegenerate at $\infty$

To define a nonlinear arrangement exhibiting intersection properties of a linear arrangement, we begin by defining a nonlinear arrangement based on a central hyperplane arrangement. We modify the approach to nonlinear arrangements given in [D2]. Consider a smooth complete intersection  $X \subset \mathbb{C}^n$  of dimension  $r$  defined by a polynomial map  $g = (g_1, \dots, g_{n-r}) : \mathbb{C}^n \rightarrow \mathbb{C}^{n-r}$ . We let  $\deg(g_i) = d_i$  and refer to  $X$  as having multidegree  $\mathbf{d} = (d_1, \dots, d_{n-r})$ . Also, consider a central hyperplane arrangement  $\mathcal{A} (= \cup H_i) \subset \mathbb{C}^p$  (for which all of the hyperplanes  $H_i$  contain 0). Let  $\varphi : \mathbb{C}^n \rightarrow \mathbb{C}^p$  be a polynomial mapping.

**Definition 1.1.** A nonlinear (affine) arrangement of smooth hypersurfaces  $A \subset X$  based on a central hyperplane arrangement  $\mathcal{A} \subset \mathbb{C}^p$  is defined by  $A = \varphi'^{-1}(\mathcal{A})$  where  $\varphi : \mathbb{C}^n \rightarrow \mathbb{C}^p$  is a polynomial map and  $\varphi' = \varphi|_X$  is transverse to  $\mathcal{A}$  (i.e.  $\varphi'$  is transverse to each flat of  $\mathcal{A}$ ).

In the special case that  $X = \mathbb{C}^n$ , we obtain a nonlinear arrangement  $A \subset \mathbb{C}^n$ . If  $A' \subset \mathbb{C}^n$  is a nonlinear arrangement and  $X$  is a smooth complete intersection which is transverse to  $A'$  (i.e. to the “nonlinear flats” of  $A'$ ), then  $A = X \cap A'$  is a nonlinear arrangement in  $X$ .

**Example 1.2.** If  $A_n \subset \mathbb{C}^n$  denotes the Boolean arrangement of coordinate hyperplanes and  $\varphi : \mathbb{C}^2 \rightarrow \mathbb{C}^n$  is a polynomial mapping whose coordinate functions are generic quadratic polynomials, let  $A = \varphi^{-1}(A_n)$ . Then  $A$  is a general position arrangement of quadrics as in figure 1 for  $n = 1, 2, 3$ .

**Example 1.3.** If in place of  $A_n$  in example 1.2, we consider the braid arrangement  $B_3 \subset \mathbb{C}^3$  and let  $\varphi : \mathbb{C}^2 \rightarrow \mathbb{C}^3$  be given by

$$\varphi(z_1, z_2) = (2z_1^2 + 6z_2^2 - 8, z_1^2 + 4z_2^2 - 5, 4z_1^2 + 7z_2^2 - 11)$$

then  $A$  consists of the three quadrics whose common intersection exhibits the triple intersection of the braid arrangement  $B_3$ . This nonlinear arrangement has 9 bounding cycles as we see in figure 3

**Example 1.4.** Again let  $A_n \subset \mathbb{C}^n$  denote the Boolean arrangement of coordinate hyperplanes and let  $\varphi : \mathbb{C} \rightarrow \mathbb{C}^n$  be a polynomial mapping whose coordinate functions are generic polynomials of degrees  $(q_1, \dots, q_n)$ . Then, set-theoretically,  $A = \varphi^{-1}(A_n)$  is a collection of

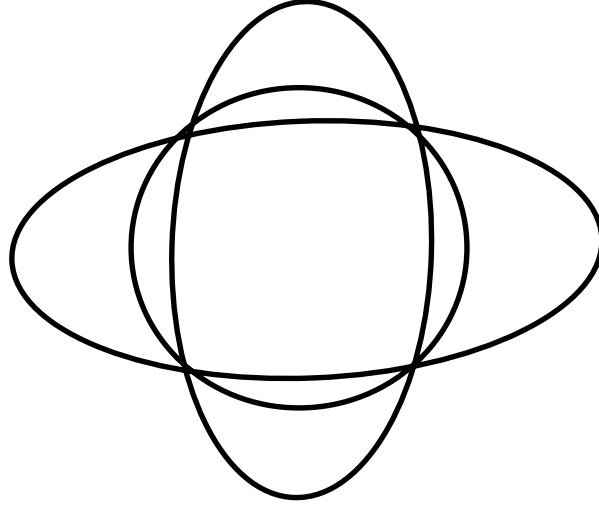


Figure 3

$q = \sum q_i$  points in  $\mathbb{C}$ , and appears identical to a general position arrangement of points in  $\mathbb{C}$ . However,  $A$  is a nonlinear general position arrangement of 0-dimensional varieties each consisting of  $q_i$  points. It is analogous to a colored braid arrangement where sets of  $q_i$  points share the same color and are indistinguishable.

**Example 1.5.** Lastly, we let  $X \subset \mathbb{C}^3$  denote a quadric surface and we consider the nonlinear arrangement  $A$  on  $X$  obtained as the intersection  $A = X \cap B_3$ . A real version of this is given in figure 4. Although  $H_1(A)$  is generated by 5 cycles, there are 6 relative bounding cycles corresponding to the 6 regions in the ellipsoid.

The conditions we have given for nonlinear arrangements are not in themselves sufficient to allow us to determine the number of bounding cycles. We must also control the behavior at infinity for both the complete intersection  $X$  and the nonlinear arrangement  $A$ .

(1.6) We do this by viewing both as the intersection of singular complete intersections and nonlinear arrangements in  $\mathbb{C}^{n+1}$  with the affine space  $\mathbb{C}^n \times \{1\}$ . We extend both polynomial mappings  $g$  and  $\varphi$  by homogenization. However, we view the homogenized maps as germs at 0. For example, from  $g = (g_1, \dots, g_{n-r}) : \mathbb{C}^n \rightarrow \mathbb{C}^{n-r}$  defining  $X$ , we define  $G = (G_1, \dots, G_{n-r}) : \mathbb{C}^{n+1}, 0 \rightarrow \mathbb{C}^{n-r}, 0$  where

$$G_i(z_1, \dots, z_{n+1}) = (z_{n+1})^{d_i} \cdot g_i(z_1/z_{n+1}, \dots, z_n/z_{n+1})$$

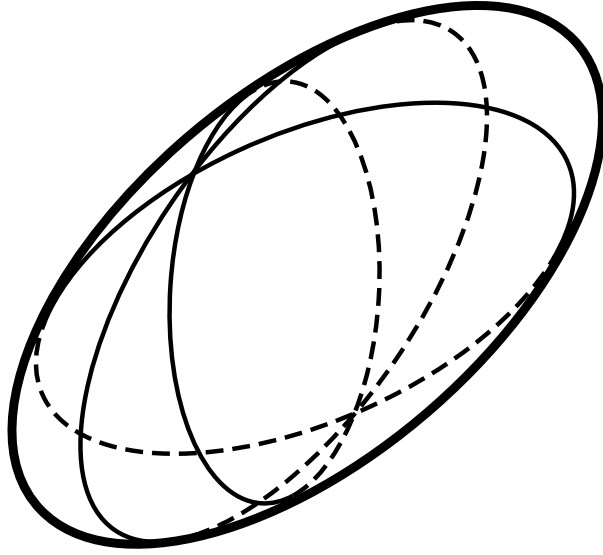


Figure 4

with  $d_i = \deg(g_i)$ . We similarly define  $\Phi : \mathbb{C}^{n+1}, 0 \rightarrow \mathbb{C}^p, 0$  from  $\varphi$ . We let  $\mathcal{X} = G^{-1}(0)$ ,  $\Phi' = \Phi|_{\mathcal{X}}$ , and  $\tilde{\mathcal{A}} = \Phi'^{-1}(\mathcal{A})$ .

The properties of  $X$  and  $A$  at infinity are given by the properties of  $\mathcal{X}$  and  $\tilde{\mathcal{A}}$ .

**Definition 1.7.** First, we say that  $X$  is smooth including  $\infty$  if both  $G$  and  $G|_{\mathbb{C}^n}$  define isolated complete intersection singularities (ICIS)  $\mathcal{X} = G^{-1}(0)$  and  $X_0 = \mathcal{X} \cap \mathbb{C}^n$  (the conditions imply that  $\mathcal{X}$  and  $\mathbb{C}^n$  are transverse off 0). Second, we say that the nonlinear arrangement  $A \subset X$  is nondegenerate at  $\infty$  if both  $\Phi' = \Phi|_{\mathcal{X}}$  and  $\Phi|_{X_0}$  are transverse to  $\tilde{\mathcal{A}}$  in a punctured neighborhood of 0.

In figure 5, We observe the relation between  $\mathcal{X}$ ,  $A$ , etc.

The results that we obtain will apply to a nonlinear arrangement  $A \subset X$  which is nondegenerate at  $\infty$  and  $X$  is smooth including  $\infty$ . Observe that if  $\tilde{\mathcal{A}} = \Phi'^{-1}(\mathcal{A})$ ,  $\mathcal{X}$ , and  $\mathbb{C}^n$  are in general position off 0, then  $A' = A \cap X \subset X$  is a nonlinear arrangement which is nondegenerate at  $\infty$  (this was the definition used in [D2]).

We should mention that we add the descriptive term “affine” in the referring to a nonlinear arrangement to distinguish from the case of a central nonlinear arrangement such as  $\tilde{\mathcal{A}} = \Phi^{-1}(\mathcal{A})$  where the defining map  $\Phi$  is only required to be transverse to  $\mathcal{A}$  off 0 (see [D1, §6]).

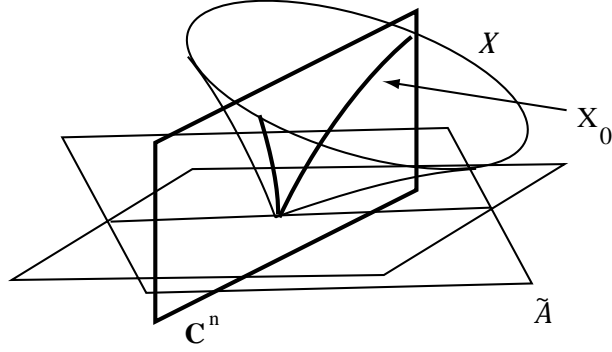


Figure 5

In order to apply deletion–restriction later to nonlinear arrangements, we shall make use of the following lemma.

**Lemma 1.8.** *Suppose  $X \subset \mathbb{C}^n$  is smooth including  $\infty$ , and  $A \subset X$  is a nonlinear arrangement nondegenerate at  $\infty$  (defined as  $\varphi^{-1}(A)$ ). Let  $K$  be a flat of  $\mathcal{A}$  and  $\mathcal{A}^K$  denote the restriction of  $\mathcal{A}$  to  $K$ . Then,  $X_K = \varphi^{-1}(K)$  is smooth including  $\infty$ , and  $A^K = \varphi^{-1}(\mathcal{A}^K) \subset X_K$  is a nonlinear arrangement nondegenerate at  $\infty$ .*

*Proof.* By assumption, both  $\Phi' : \mathcal{X} \rightarrow \mathbb{C}^p$  and  $\Phi'|_{X_0}$  are transverse to the central hyperplane arrangement  $\mathcal{A} \subset \mathbb{C}^p$  in a punctured neighborhood of 0. In particular, they are transverse to  $K$  in a punctured neighborhood of 0. Then, by a straightforward fiber square argument,  $\mathcal{X}_K = \Phi'^{-1}(K)$  and  $X_{0K} = \Phi'^{-1}(K) \cap X_0$  are ICIS, and both  $\Phi'|_{\mathcal{X}_K} : \mathcal{X}_K \rightarrow K$  and  $\Phi'|_{X_{0K}} : X_{0K} \rightarrow K$  are transverse to  $\mathcal{A}^K$  in a punctured neighborhood of 0. Hence,  $A^K \subset X_K$  is a nonlinear arrangement nondegenerate at  $\infty$ . Q.E.D.

*Reduction from Global to Local Properties:*

Suppose  $A \subset X$  is a nonlinear affine arrangement nondegenerate at  $\infty$  and that  $X$  is smooth including  $\infty$ . We construct as in (1.6) the associated homogeneous objects  $\mathcal{X}$ , etc. There is a basic relation between the local properties of the homogeneous objects and the corresponding affine ones. First, both  $\mathcal{X}$  and  $\tilde{\mathcal{A}}$  are transverse to  $\mathbb{C}^n$  off 0. Second, since  $\mathbb{C}^n$  is transverse to  $\mathcal{X}$  off 0, it follows that  $\mathbb{C}^n \times \{t\}$  is transverse to  $\mathcal{X}$  and  $X = \mathcal{X} \cap (\mathbb{C}^n \times \{1\})$  is the smooth complete intersection in  $\mathbb{C}^n \times \{1\} \simeq \mathbb{C}^n$ . We let  $X_t = \mathcal{X} \cap (\mathbb{C}^n \times \{t\})$ . Likewise,  $A = \tilde{\mathcal{A}} \cap (\mathbb{C}^n \times \{1\})$  is the nonlinear affine arrangement in  $X$ . Third,  $\varphi = \Phi|_{\mathbb{C}^n \times \{1\}}$ .

Then, by a theorem of Hamm [Ha], for a sufficiently small ball  $B_\epsilon$  and  $0 < |t| \ll \epsilon$ ,  $X_t \cap B_\epsilon$  is the Milnor fiber of the ICIS  $X_0$  and is homotopy equivalent to a bouquet of spheres of (real) dimension  $r$ . The number of such spheres is the Milnor number, which we denote by  $\mu(X_0)$ .

Also, by [D1, §7] (and see [D3]), for  $\epsilon$  sufficiently small and  $0 < |t| \ll \epsilon$ ,  $X_t \cap \tilde{\mathcal{A}} \cap B_\epsilon$  is the singular Milnor fiber of the nonisolated complete intersection singularity  $A_0, 0$  viewed as a nonlinear section of  $\{0\} \times \mathcal{A}, 0 \subset \mathbb{C}^{n-r+1+p}, 0$ . Using a theorem of Lê [Lê1], it is also proven that  $X_t \cap \tilde{\mathcal{A}} \cap B_\epsilon$  is homotopy equivalent to a bouquet of spheres of (real) dimension  $r - 1$ . Then,  $X_t \cap \tilde{\mathcal{A}} \cap B_\epsilon$  is called the *singular Milnor fiber* of  $A_0$ , and the number of such spheres is the singular Milnor number, denoted by  $\mu(A_0)$  ([DM, §4], [D1, §7] and [D3]). Strictly speaking, these (singular) Milnor fibers are associated to the mappings  $g' = (g, z_{n+1})$  and  $(g', \Phi)$  defining  $X_0$  and  $A_0$  as an ICIS or a nonlinear section of a complete intersection; however, by [D1, §6, 7] the description given here is consistent with the definition for the mappings.

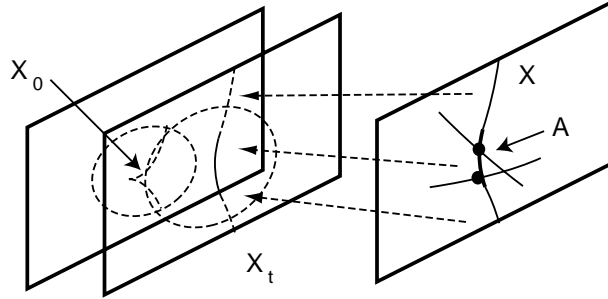


Figure 6

The global affine spaces  $X$  and  $A$  are related to the (singular) Milnor fibers by the following result, which is a slight variant of Prop. 2.5 in [D2], but whose proof is virtually identical.

**Proposition 1.9.** *Suppose  $A \subset X$  is a nonlinear (affine) arrangement nondegenerate at  $\infty$  (with  $X$  smooth including  $\infty$ ). Then,  $X$  is homeomorphic to the Milnor fiber  $X_t \cap B_\epsilon$  of  $X_0$  via a homeomorphism  $\psi$  which can be chosen to send  $A$  to  $X_t \cap \tilde{\mathcal{A}} \cap B_\epsilon$  (see fig. 6).*

*Number of Bounding Cycles as Relative Singular Milnor Numbers:*

In light of the preceding discussion of the relation between nonlinear arrangements and singular Milnor fibers, we define the number of bounding cycles in terms of (relative) homology of the nonlinear arrangement.



**Definition 1.10.** If the nonlinear arrangement  $A \subset \mathbb{C}^n$  is nondegenerate at  $\infty$ , then the *number of bounding cycles* is defined to be  $\dim H_{n-1}(A)$ . If  $A \subset X$  is a nonlinear (affine) arrangement nondegenerate at  $\infty$  (with  $X$  smooth including  $\infty$ ), then the number of *relative bounding cycles* is defined to be  $\dim H_r(X, A)$ .

We observe that if  $X = \mathbb{C}^n$ , then the number of relative bounding cycles for  $A$  is the same as the number of bounding cycles. Because the relative Milnor fiber  $(X_t \cap B_\epsilon, X_t \cap \tilde{\mathcal{A}} \cap B_\epsilon)$  is a relative CW-complex of dimension  $r$ , by the homotopy properties of (singular) Milnor fibers we see that

$$\begin{aligned} \dim H_r(X_t \cap B_\epsilon, X_t \cap \tilde{\mathcal{A}} \cap B_\epsilon) \\ = \dim H_r(X_t \cap B_\epsilon) + \dim H_{r-1}(X_t \cap \tilde{\mathcal{A}} \cap B_\epsilon) \end{aligned}$$

By proposition 1.9 and the exact sequence of a pair, this implies for the affine spaces that  $H_k(X, A)$  is only nonzero when  $k = r$  and

$$(1.11) \quad \dim H_r(X, A) = \dim H_r(X) + \dim H_{r-1}(A)$$

We shall refer to  $\dim H_r(X_t \cap B_\epsilon, X_t \cap \tilde{\mathcal{A}} \cap B_\epsilon)$  as the relative singular Milnor number of  $A_0$ . Via proposition 1.9 and 1.11, we can summarize the discussion by

$$(1.12) \quad \begin{aligned} &\text{the number of relative bounding cycles for } (X, A) = \\ &\text{the relative singular Milnor number of } (X_0, A_0) \end{aligned}$$

**Remark 1.13.** The singular Milnor numbers can be explicitly computed in the case that  $\mathcal{A}$  is a free arrangement [D1, §6], then  $A_0$  is called an *almost free arrangement* and  $\mu(A_0)$  can be computed as the length of a determinantal module, see [DM, thms 5, 6] and [D1, §4]. This is further extended in [D1, §7, 8] to *almost free complete intersections*, intersections of almost free divisors which are the transverse off 0. This includes nonlinear arrangements such as  $A_0$ .

It was the formulas for almost free nonlinear arrangements  $A_0$  which suggested the existence and form for a general formula given in §5.

## §2. Higher Multiplicities of Central Arrangements

A general formula for the number of bounding cycles must be expressed in terms on intrinsic invariants of arrangements. We recall just such a set of intrinsic geometric invariants of central arrangements, viewed as nonisolated singularities. These are the higher multiplicities.

For the case of isolated hypersurface singularities, Teissier [Te] introduced a series of higher multiplicities, namely the  $\mu_i$  appearing in his  $\mu^*$ -sequence  $\mu^* = (\mu_0, \dots, \mu_n)$ . Specifically, given  $f_0 : \mathbb{C}^n, 0 \rightarrow \mathbb{C}, 0$ , if  $\Pi$  is a generic  $s$ -dimensional subspace in  $\mathbb{C}^n$  then  $f_0|_{\Pi}$  has an isolated singularity and Teissier defines  $\mu_s(f_0) = \mu(f_0|_{\Pi})$ , where  $\mu(\cdot)$  denotes the usual Milnor number. This was extended to arbitrary singularities  $(V, 0)$  by Lê and Teissier [LêT] considering instead generic projections  $V, 0 \rightarrow \Pi, 0$  for linear subspaces  $\Pi$  of varying dimensions. They consider the Euler characteristics of Milnor fibers of such projections.

In [D1, §4], we considered higher multiplicities for nonisolated complete intersection singularities  $V, 0 \subset \mathbb{C}^n, 0$  using the analogue of Teissier's original definition. A Zariski open subset of  $s$ -dimensional subspaces  $\Pi \subset \mathbb{C}^n$  are (geometrically) transverse to  $V$  off 0. We view the inclusion  $i : \Pi \rightarrow \mathbb{C}^n$  as a section of  $V$ . For simplicity we assume  $P = \mathbb{C}^s$  so that we have a map germ  $i : \mathbb{C}^s, 0 \rightarrow \mathbb{C}^n, 0$  which is then transverse to  $V$  off 0. By the  $s$ -th higher multiplicity  $\mu_s(V)$  we mean the singular Milnor number of the generic nonlinear section  $i$  (and  $\mu_0(V) \stackrel{\text{def}}{=} 1$ ). For a central arrangement  $\mathcal{A} \subset \mathbb{C}^p$ , if  $i_t$  is a perturbation of  $i$  which is transverse to  $\mathcal{A}$ , then the affine arrangement  $i_t(\mathbb{C}^s) \cap \mathcal{A}$  is homotopy equivalent to a bouquet of  $s-1$ -spheres and  $\mu_s(\mathcal{A})$  counts the number of such spheres.

We summarize the main properties on the higher multiplicities of central hyperplane arrangements.

**Proposition 2.1.** *Suppose that  $\mathcal{A} \subset \mathbb{C}^p$  is a central arrangement.*

- (1) *If  $r = r(\mathcal{A})$  is the rank of  $\mathcal{A} = \cup H_i$  ( $= \text{codim}(\cap H_i)$ ), then  $\mu_k(\mathcal{A}) = 0$  if  $k \geq r$  and*

$$\mu_{r-1}(\mathcal{A}) = |\mu_{\text{Möb}}(\mathcal{A})|$$

*( $\mu_{\text{Möb}}(\mathcal{A})$  denotes the Möbius function of the lattice  $L(\mathcal{A})$ , [OT, Chap. 2]);*

- (2) *If  $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$  is a deletion-restriction triple, then*

$$\mu_k(\mathcal{A}) = \mu_k(\mathcal{A}') + \mu_{k-1}(\mathcal{A}'')$$

- (3) *The  $k$ -th Betti number of the complement  $M(\mathcal{A}) = \mathbb{C}^p \setminus \mathcal{A}$  is given by*

$$b_k(M(\mathcal{A})) = \mu_k(\mathcal{A}) + \mu_{k-1}(\mathcal{A})$$

*(in particular,  $b_r(M(\mathcal{A})) = \mu_{r-1}(\mathcal{A})$  as  $\mu_r(\mathcal{A}) = 0$ ).*

- (4) *If  $\mathcal{A}$  is a free arrangement, then*

$$\mu_k(\mathcal{A}) = \sigma_k(\text{exp}'(\mathcal{A}))$$

If  $\exp(\mathcal{A}) = (e_0, e_1, \dots, e_{p-1})$  with  $e_0 = 1$ , then  $\exp'(\mathcal{A}) = (e_1, \dots, e_{p-1})$  and  $\sigma_k(\cdot)$  denotes the  $k$ -th elementary symmetric function.

- (5) If  $\mathcal{A}$  is the complexification of a real, essential arrangement  $\mathcal{A}_{\mathbb{R}}$ , then the  $\beta$  invariant of Crapo is given by

$$\beta(\mathcal{A}) = \mu_{p-1}(\mathcal{A})$$

*Proof.* The proofs of these results essentially follow from [D1, §5]. 1), 3), and 4) are respectively Proposition 5.6, Lemma 5.6, and Proposition 5.2 of [D1]. 5) follows from 1) and the equality of  $\beta(\mathcal{A})$  and  $\mu_{\text{Möb}}(\mathcal{A})$ . Lastly, for 2) if  $\Pi$  is a generic  $k$ -dimensional subspace, then  $(\mathcal{A} \cap \Pi, \mathcal{A}' \cap \Pi, \mathcal{A}'' \cap \Pi)$  is still a deletion–restriction triple. Then, if  $k + 1 = \dim(\Pi) > r(\mathcal{A})$  then  $\mu_k(\mathcal{A})$ ,  $\mu_k(\mathcal{A}')$ , and  $\mu_{k-1}(\mathcal{A}'')$  are zero. If  $k < r(\mathcal{A})$ , then by 1) these multiplicities are  $|\mu_{\text{Möb}}(\mathcal{A} \cap \Pi)|$ ,  $|\mu_{\text{Möb}}(\mathcal{A}' \cap \Pi)|$ , and  $|\mu_{k-1}(\mathcal{A}'' \cap \Pi)|$ . Then, the result follows from Corollary 2.3.12 of [OT] (if  $H \cap \Pi$  is not a separator, then  $r(\mathcal{A}' \cap \Pi) < k + 1$  so  $\mu_k(\mathcal{A}') = 0$ ). Q.E.D.

We give several corollaries.

**Corollary 2.2.** For a central hyperplane arrangement  $\mathcal{A} \subset \mathbb{C}^p$ , the Poincaré polynomial of the complement  $M(\mathcal{A})$  is given by

$$P(\mathcal{A}, t) = (1 + t) \cdot \mu(\mathcal{A}, t) \quad \text{where} \quad \mu(\mathcal{A}, t) = \sum_{j=0}^{p-1} \mu_j(\mathcal{A}) t^j$$

is the “multiplicity polynomial” of  $\mathcal{A}$ .

We should note that in [OT],  $P(\mathcal{A}, t)$  is equivalently given by  $\pi(\mathcal{A}, t)$ .

As a second corollary we obtain a formula for the multiplicities of a “product of arrangements”  $\mathcal{A} \subset \mathbb{C}^{p_1}$  and  $\mathcal{B} \subset \mathbb{C}^{p_2}$ . This is the arrangement  $(\mathcal{A} \times \mathbb{C}^{p_2}) \cup (\mathbb{C}^{p_1} \times \mathcal{B}) \subset \mathbb{C}^p$ , where  $p = p_1 + p_2$ . This same construction appears in other situations in singularity theory where the term “product” is misleading, so we refer to it more generally as the “product union” of  $\mathcal{A}$  and  $\mathcal{B}$ , and denote it by  $\mathcal{A} \boxtimes \mathcal{B}$  [D1, §3]. Let

$$\lambda_k(\mathcal{A}, \mathcal{B}) = \sum_{j=0}^k \mu_j(\mathcal{A}) \mu_{k-j}(\mathcal{B}).$$

**Corollary 2.3.** For a central hyperplane arrangements  $\mathcal{A}$  and  $\mathcal{B}$

$$(2.4) \quad \mu_k(\mathcal{A} \boxtimes \mathcal{B}) = \lambda_k(\mathcal{A}, \mathcal{B}) + \lambda_{k-1}(\mathcal{A}, \mathcal{B})$$

*Proof.* By [OT, Lemma 2.3.3]

$$P(\mathcal{A} \bowtie \mathcal{B}, t) = P(\mathcal{A}, t) \cdot P(\mathcal{B}, t)$$

Then, using Corollary 2.2 we conclude

$$(2.5) \quad \mu(\mathcal{A} \bowtie \mathcal{B}, t) = (1+t)\mu(\mathcal{A}, t) \cdot \mu(\mathcal{B}, t)$$

Equating coefficients of  $t^k$  in (2.5) yields the corollary Q.E.D.

**Remark 2.6.** The expression in Corollary 2.3 does not obviously depend upon  $\mathcal{A}$  nor  $\mathcal{B}$  being arrangements. This suggests the conjecture that Formula 2.4 is valid for arbitrary germs of hypersurfaces  $\mathcal{A}$  and  $\mathcal{B}$ .

### §3. Nonlinear Deletion–Restriction

We consider a central arrangement  $\mathcal{A} \subset \mathbb{C}^p$  with  $H$  a hyperplane in  $\mathcal{A}$ . Let  $(H \subset \mathcal{A}, \mathcal{A}', \mathcal{A}'')$  be a deletion–restriction triple for the hyperplane  $H$  [OT, def. 1.2.14]. Recall it consists of arrangements  $\mathcal{A}' = \{H' \in \mathcal{A} : H' \neq H\}$ , and  $\mathcal{A}'' = \mathcal{A}^H (= H \cap (\cup H'))$ , where the union is over  $H' \in \mathcal{A}'$ .

**Definition 3.1.** Let  $A \subset X$  be a nonlinear arrangement nondegenerate at  $\infty$  defined as  $\varphi'^{-1}(\mathcal{A})$  for  $\varphi' = \varphi|_X$  (with  $X$  smooth including  $\infty$ ). The associated *nonlinear deletion–restriction triple*  $(H' \subset A, A', A'')$  consists of the smooth hypersurface  $H' = X_H$ , and the nonlinear arrangements  $A, A' = \varphi'^{-1}(\mathcal{A}')$ , and  $A'' = \varphi'^{-1}(\mathcal{A}'')$ .

By proposition 1.8,  $H'$  is smooth including  $\infty$  and both  $A' \subset X$  and  $A'' \subset H'$  are nondegenerate at  $\infty$ .

If  $\chi(Y)$  denotes the Euler characteristic of  $Y$ , then nonlinear deletion–restriction takes the following simple form.

**Proposition 3.2** (Nonlinear Deletion–Restriction). *For the nonlinear deletion–restriction triple  $(H' \subset A, A', A'')$ ,*

$$\chi(X, A) = \chi(X, A') - \chi(H', A'')$$

*Proof.* This is a simple application of Euler characteristic arguments for exact sequences. As  $A'' = A' \cap H'$  and  $A = A' \cup H'$ , Mayer–Vietoris implies

$$(3.3) \quad \chi(A) = \chi(H') + \chi(A') - \chi(A'')$$

Subtracting both sides of (3.3) from  $\chi(X)$ , and using that for  $Y \subset X$ ,  $\chi(X, Y) = \chi(X) - \chi(Y)$ , we obtain the conclusion. Q.E.D.

Normally, Proposition 3.2 would not yield strong conclusions. However, in our case all of the arrangements are nondegenerate at  $\infty$ . We recall by proposition 1.9 and the discussion following that  $X$  is homeomorphic to the Milnor fiber of an ICIS of dimension  $r$ , and  $A$  is homeomorphic to the singular Milnor fiber for a nonlinear section of a (nonisolated) complete intersection. Then,  $(X, A)$  is a relative CW-complex of dimension  $r$ , and  $X$  and  $A$  are homotopy equivalent to bouquets of spheres of dimensions  $r$ , respectively  $r - 1$ . Hence, by (1.11)

$$\begin{aligned}
 \chi(X, A) &= (-1)^r \dim H_r(X, A) \\
 (3.4) \quad &= (-1)^r (\dim H_r(X) + \dim H_{r-1}(A)) \\
 &= (-1)^r (\mu(X_0) + \mu(A_0))
 \end{aligned}$$

(here  $\mu(X_0)$  and  $\mu(A_0)$  denote the (singular) Milnor numbers of the (singular) Milnor fibers).

Thus, nonlinear deletion–restriction (3.2) takes the form

$$(3.5) \quad \dim H_r(X, A) = \dim H_r(X, A') + \dim H_{r-1}(H', A'')$$

Our next goal is to find a formula for  $\dim H_r(X, A)$  which satisfies (3.5).

**Remark.** Even if we only wanted a formula for the number of bounding cycles for a nonlinear arrangement in  $\mathbb{C}^n$ , we see that deletion–restriction leads us to consider via (3.5) the relative number of bounding cycles of  $A''$  on the nonlinear hypersurface  $H'$ . Hence, it is really necessary to establish a general result of the form we obtain.

#### §4. Functional Equations for the Number of Bounding Cycles

The formula for the number of relative bounding cycles is a formula for  $H^r(X, A)$  which must satisfy nonlinear deletion–restriction in the form (3.5). At this point *we restrict to the case where all of the coordinate functions of  $\Phi$  are homogeneous of the same degree  $q$*  so that  $A$  is a nonlinear arrangement of  $q$ -tic hypersurfaces, as will be any nonlinear arrangement obtained by deletion or restriction. In the special case when  $\mathcal{A}$  consists of a single hyperplane  $H$ , we also have  $A$  is the Milnor fiber of the homogeneous ICIS  $\mathcal{X} \cap \Phi^{-1}(H) \cap \mathbb{C}^n$  which has multidegree  $\mathbf{d} = (d_1, \dots, d_{n-r}, q)$ . Thus, as in (3.4)  $\dim H_r(X, A)$  is the sum of two Milnor numbers of homogeneous ICIS, and hence is given by the formulas of Greuel–Hamm [GH] and Giusti [Gi] which only involve the multidegree. We denote these formulas for Milnor numbers by  $\mu(\mathbf{d})$  and  $\mu(\mathbf{d}, q)$ .

Also, in the case when  $\mathcal{A}$  is a free arrangement and  $\Phi$  is homogeneous, we may apply the formula for the singular Milnor number  $\mu(A)$

in [D1, Thm 2] together with the formula from [DM, Thm 5], together with proposition 2.3 of [D4] to conclude that for  $\mathcal{A}$  fixed,  $\mu(A)$  only depends on the multidegree  $\mathbf{d} = (d_1, \dots, d_{n-r})$ , and the degree  $q$ . Thus, we seek a formula for  $\dim H_r(X, A)$  in the form  $p(\mathbf{d}, q, \mathcal{A})$  which satisfies the equation (3.5) so that when  $\mathcal{A}$  is a single hyperplane, it becomes the sum of the Milnor numbers for the pair of ICIS  $(X, A)$ . These equations become the following functional equations.

(4.1) *Functional Equations for a Nonlinear Deletion–Restriction Triple:*

$$(H \subset \mathcal{A}, \mathcal{A}', \mathcal{A}'')$$

(1)

$$p(\mathbf{d}, q, \mathcal{A}) = p(\mathbf{d}, q, \mathcal{A}') + p((\mathbf{d}, q), q, \mathcal{A}'')$$

(2)

$$p(\mathbf{d}, q, \{H\}) = \mu(\mathbf{d}) + \mu(\mathbf{d}, q)$$

**Remark.** To reduce excessive notation, it will be understood in the functional equations (4.1) that the ambient space for the nonlinear arrangements is  $\mathbb{C}^n$ .

**Proposition 4.2.** *Suppose  $p(\mathbf{d}, q, \mathcal{A})$  satisfies the functional equations (4.1) for all deletion–restriction triples  $(H \subset \mathcal{A}, \mathcal{A}', \mathcal{A}'')$ . If  $X$  is a homogeneous hypersurface of multidegree  $\mathbf{d}$ , smooth including  $\infty$  and  $A \subset X$  is a nonlinear arrangement of smooth  $q$ -tic hypersurfaces non-degenerate at  $\infty$ , then*

$$\dim H_r(X, A) = p(\mathbf{d}, q, \mathcal{A})$$

*Proof.* This is proven by induction on the number  $|\mathcal{A}|$  of hyperplanes in  $\mathcal{A}$ . For one hyperplane, it follows by (4.1-2). Then, by the induction hypothesis, if it holds for arrangements  $\mathcal{A}'$  with  $|\mathcal{A}'| < m$  and  $|\mathcal{A}| = m$ , then by (4.1-1) and (3.5) we obtain the result for  $\mathcal{A}$ . Q.E.D.

## §5. Formula for the Number of Relative Bounding Cycles

To find a candidate for  $p(\mathbf{d}, q, \mathcal{A})$ , we examine special cases obtained in [D2] for the case that  $\mathcal{A}$  is a free arrangement. The special cases compute the singular Milnor number of central nonlinear arrangements based on free arrangements. The (relative) singular Milnor number computes the the number of (relative) bounding cycles for the associated affine nonlinear arrangement.

First for hyperplane arrangements, we recall ([D1, §5] or [D3, §7]) that if  $\Phi : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^p$  and  $\Phi|_{(\mathbb{C}^n \times \{0\})}$  are linear embeddings transverse

to  $\mathcal{A}$  off 0, then  $\tilde{\mathcal{A}} = \Phi^{-1}(\mathcal{A})$  is called an almost free arrangement and  $A = \tilde{\mathcal{A}} \cap (\mathbb{C}^n \times \{1\})$  is called an *almost free affine arrangement* (based on  $\mathcal{A}$ ). We also refer to  $A$  as being  $\mathcal{A}$ -generic. For example, if  $\mathcal{A}$  is a Boolean arrangement, then an affine  $\mathcal{A}$ -generic arrangement is a general position arrangement.

The almost free affine arrangement  $A$  is the singular Milnor fiber of a generic hyperplane section of the almost free arrangement  $\tilde{\mathcal{A}} = \Phi^{-1}(\mathcal{A})$ . The singular Milnor number  $\mu(\tilde{\mathcal{A}})$  also gives the higher multiplicity  $\mu_n(\mathcal{A})$  [D1, §4,5]. Also, for an almost free arrangement  $A$  based on  $\mathcal{A}$ ,  $\mu_k(\mathcal{A}) = \sigma_k(\exp'(\mathcal{A}))$  where  $\exp'(\mathcal{A}) = (e_1, \dots, e_{p-1})$  and  $\sigma_k(\mathbf{x})$  denotes the  $k$ -th elementary symmetric function in  $\mathbf{x} = (x_1, \dots, x_{p-1})$ .

**Notation.** In the formulas that follow, in addition to the elementary symmetric functions  $\sigma_k(\mathbf{x})$ , we shall also need the collection of related functions  $s_k(\mathbf{x})$ . Here  $s_k(\mathbf{x})$  is defined to be the polynomial defined as the sum of all monomials of degree  $k$  in  $\mathbf{x}$  (here  $s_0(\mathbf{x}) \stackrel{\text{def}}{=} 1$ ). These functions naturally complement the elementary symmetric functions, have analogous expansions as well as other properties listed in [D4, §2]. In using these functions we will have occasion to evaluate  $\sigma_k(\mathbf{x})$  where  $x_j = a$  for say the last  $\ell$  values of  $j$ . We indicate this by  $\sigma_k(x_1, \dots, x_{p-\ell}, a^\ell)$ . We may do this for several different  $a_i$ , and as well for the functions  $s_k$ .

Two special cases of the general formula we seek are given by the following.

**Proposition 5.1** ([D1, prop. 6.12]). *Let  $A$  be an  $\mathcal{A}$ -generic affine nonlinear arrangement of hypersurfaces each of degree  $q$  (with  $\mathcal{A}$  free). Then,*

$$(5.2) \text{ the number of bounding cycles of } A = \sigma_n((q-1)^n, qe_1, \dots, qe_{p-1})$$

(where again  $\exp'(\mathcal{A}) = (e_1, \dots, e_{p-1})$ ).

In [D1], we used a special form of nonlinear arrangement  $A = X \cap A'$  where  $X$  is the transverse to the nonlinear arrangement  $A'$  including points at  $\infty$  (see also [D2, def. 2.6]).

**Proposition 5.3** ([D1, Theorem 8.19]). *Suppose that  $A'$  is an  $\mathcal{A}$ -generic affine hyperplane arrangement with  $\mathcal{A}$  free. Let  $A = X \cap A'$  where  $X$  is smooth of multidegree  $\mathbf{d}$  including  $\infty$  and transverse to  $A'$  including  $\infty$ . Then*

$$(5.4) \quad \begin{aligned} & \text{the number of relative bounding cycles of } (X, A) \\ &= d \cdot \left( \sum_{j=0}^r s_j(\mathbf{d} - \mathbf{1}) \mu_{r-j}(\mathcal{A}) \right) \end{aligned}$$

where  $d = \prod_{i=1}^{n-r} d_i$  and  $\mathbf{d} - \mathbf{1} = (d_1 - 1, \dots, d_{n-r} - 1)$ .

Comparing these two results we first notice that (5.4) is given in terms of the multidegree  $\mathbf{d}$  and the higher multiplicities  $\mu_j(\mathcal{A})$ , while that (5.2) is not. Second, in the special case of a smooth hypersurface  $X$ , (5.4) can be reexpressed as follows.

**Corollary 5.5.** *Suppose  $A'$  is an  $\mathcal{A}$ -generic affine arrangement (with  $\mathcal{A}$  free), and that  $X$  is an  $r$ -dimensional smooth hypersurface of degree  $d + 1$  which is smooth and transverse to  $A'$  including  $\infty$ . Then, for  $A = X \cap A'$*

$$(5.6) \quad \text{the number of relative bounding cycles of } (X, A) = d^n \cdot P(\mathcal{A}, d^{-1})$$

**Remark.** It follows from results of Orlik-Terao [OT2] that when  $X$  is a homogeneous hypersurface, the relative Euler characteristic equals the RHS of (5.6) for *arbitrary arrangements*  $\mathcal{A}$ . This suggests that Proposition 5.3 should hold without the condition on  $\mathcal{A}$ . To compare (5.2) and (5.4) we first restate (5.2) in a form involving the higher multiplicities as follows.

**Proposition 5.7.** *Let  $A$  be an  $\mathcal{A}$ -generic affine nonlinear arrangement of hypersurfaces each of degree  $q$  (with  $\mathcal{A}$  free). Then,*

$$(5.8) \quad \begin{aligned} & \text{the number of bounding cycles of } A \\ &= \sum_{j=0}^n \binom{n}{j} (q-1)^j q^{n-j} \mu_{n-j}(\mathcal{A}) \end{aligned}$$

*Proof.* Using properties of elementary symmetric functions, we expand

$$(5.9) \quad \begin{aligned} & \sigma_n((q-1)^n, qe_1, \dots, qe_{p-1}) \\ &= \sum_{j=0}^n \sigma_j((q-1)^n) \sigma_{n-j}(qe_1, \dots, qe_{p-1}) \end{aligned}$$

Also,

$$(5.10) \quad \sigma_j((q-1)^n) = (q-1)^j \sigma_j(1^n) = (q-1)^j \binom{n}{j}$$

Similarly,



$$(5.11) \quad \sigma_{n-j}(qe_1, \dots, qe_{p-1}) = q^{n-j} \sigma_{n-j}(e_1, \dots, e_{p-1}) = q^{n-j} \mu_{n-j}(\mathcal{A})$$

Here the last equality follows from (2.1-4) as  $\mathcal{A}$  is a free arrangement. Substituting (5.10) and (5.11) into (5.9) gives the result. Q.E.D.

The formulas in Proposition 5.3 and 5.8 show a greater resemblance if we observe that by properties of the functions  $s_j$  (see [D1, §2]),

$$\binom{n}{j} (q-1)^j = s_j((q-1)^{n-j-1})$$

so that (5.8) can be written

$$(5.12) \quad \sum_{j=0}^n s_j((q-1)^{n-j-1}) q^{n-j} \mu_{n-j}(A')$$

Then, the form of (5.4) and (5.12) suggest the following candidate as a general formula.

$$(5.13) \quad p(\mathbf{d}, q, \mathcal{A}) = d \cdot \left( \sum_{j=0}^r s_j(\mathbf{d} - \mathbf{1}, (q-1)^{r-j+1}) \cdot q^{r-j} \mu_{r-j}(\mathcal{A}) \right)$$

where  $d = \prod_{i=1}^{n-r} d_i$  and  $\mathbf{d} - \mathbf{1} = (d_1 - 1, \dots, d_{n-r} - 1)$ .

We shall show that this is correct.

**Theorem 1.** *Suppose  $X \subset \mathbb{C}^n$  is an  $r$ -dimensional complete intersection of multidegree  $\mathbf{d}$ , which is smooth including  $\infty$ . Let  $A \subset X$  be a nonlinear arrangement of smooth  $q$ -tic hypersurfaces based on central arrangement  $\mathcal{A}$  and nondegenerate at  $\infty$ . Then,*

$$\begin{aligned} & \text{the number of relative bounding cycles of } (X, A) \\ & = p(\mathbf{d}, q, \mathcal{A}) \text{ given by (5.13)} \end{aligned}$$

Because (5.4) and (5.8) are special cases of this result, we obtain as a corollary.

**Corollary 2.** *In the special cases of propositions 5.3 or 5.7, except that we allow  $A \subset \mathbb{C}^n$  to be a nonlinear arrangement based on any central arrangement  $\mathcal{A}$ , then the number of bounding cycles of  $A$ , respectively the number relative bounding cycles of  $(X, A)$ , is given by (5.8), respectively (5.4).*

Also, because the number of relative bounding cycles is also a relative singular Milnor number, we can also deduce as a corollary both the singular Milnor number and higher multiplicities.

**Corollary 3.** *Let  $X_0$  be a homogeneous  $r$ -dimensional ICIS of multidegree  $\mathbf{d}$ . Also, let  $A_0 = \Phi'^{-1}(\mathcal{A}) \subset X_0$  be a nonlinear central arrangement consisting of hypersurfaces of degree  $q$ , where  $\Phi' : X_0 \rightarrow \mathbb{C}^p$  is transverse to  $\mathcal{A}$  off 0. Then,*

(1)

$$\text{the singular Milnor number } \mu(A_0) = p(\mathbf{d}, q, \mathcal{A}) - \mu(X_0)$$

(2) *Likewise, the  $k$ -th higher multiplicity is given by*

$$\mu_k(A_0) = p((\mathbf{d}, 1^{n-k}), q, \mathcal{A}) - \mu_k(X_0)$$

**Example 5.14.** We return to the examples (1.2) and (1.3) of nonlinear arrangements of quadrics  $A \subset \mathbb{C}^2$  based on a central arrangement  $\mathcal{A}$ . By Corollary 2,

$$(5.15) \quad \begin{aligned} \text{number of bounding cycles of } A &= 1 + 4(\mu_1(\mathcal{A}) + \mu_2(\mathcal{A})) \\ &= 1 + 4b_2(\mathcal{A}) \end{aligned}$$

where  $b_2(\mathcal{A})$  denotes the second Betti number of  $M(\mathcal{A})$ . For  $\mathcal{A}$  the Boolean arrangement  $A_p \subset \mathbb{C}^p$ ,  $\mu_k(A_p) = \binom{p-1}{k}$  where  $\binom{p-1}{k} = 0$  if  $k > p-1$ . After simplifying (5.15), we obtain for general position arrangements of  $p$  quadrics in general position in  $\mathbb{C}^2$ ,  $1 + 4\binom{p}{2}$  bounding cycles. This yields the numbers 1, 5, and 13 for the first three cases in Fig. 1 (providing an alternate formula to Corollary 5.1).

For example (1.3), we have by (4) of proposition 2.1, for the braid arrangement  $B_p \subset \mathbb{C}^p$ ,  $\mu_k(B_p) = \sigma_k(2, \dots, p-1)$ . So for a nonlinear braid arrangement in example (1.3), we have by (5.15) the number of bounding cycles equals  $1 + 4\binom{3}{2} = 9$ .

**Example 5.16.** Second, consider as in example (1.5) a nonlinear hyperplane braid arrangement  $A$  on the hypersurface  $X$  of degree  $d$  in  $\mathbb{C}^n$ . By Corollary 2 and Proposition 5.3, the number of relative bounding cycles equals

$$(5.17) \quad \begin{aligned} d((d-1)^{n-1} + (d-1)^{n-2} \cdot \sigma_1(2, \dots, n-1) + \dots + \\ (d-1) \cdot \sigma_{n-2}(2, \dots, n-1) + \sigma_{n-1}(2, \dots, n-1) \end{aligned}$$

For example, on  $\mathbb{C}^3$  when  $d = 2$ , we obtain  $2(1^2 + 1 \cdot 2 + 0) = 6$  relative bounding cycles as shown in Fig. 4.

## §6. Proof of the General Formula

We note that Theorem 1 is an immediate consequence of Proposition 4.2, provided we can show that  $p(\mathbf{d}, q, \mathcal{A})$  satisfies the functional equations (4.1).

For the first functional equation, we may write  $p(\mathbf{d}, q, \mathcal{A}') + p(\mathbf{d}, q, \mathcal{A}'')$  as

$$(6.1) \quad d \cdot \left( \sum_{j=0}^r s_j(\mathbf{d} - \mathbf{1}, (q-1)^{r-j+1}) \cdot q^{r-j} \mu_{r-j}(\mathcal{A}') \right) \\ + d \cdot q \cdot \left( \sum_{j=0}^{r-1} s_j((\mathbf{d}, q) - \mathbf{1}, (q-1)^{r-1-j+1}) \cdot q^{r-1-j} \mu_{r-1-j}(\mathcal{A}'') \right)$$

The first sum can be written

$$(6.2) \quad d \cdot s_r(\mathbf{d} - \mathbf{1}, q-1) \mu_0(\mathcal{A}') \\ + d \cdot \left( \sum_{j=0}^{r-1} s_j(\mathbf{d} - \mathbf{1}, (q-1)^{r-j+1}) \cdot q^{r-j} \mu_{r-j}(\mathcal{A}') \right)$$

In the second sum, we see

$$(6.3) \quad s_j((\mathbf{d}, q) - \mathbf{1}, (q-1)^{r-1-j+1}) = s_j(\mathbf{d} - \mathbf{1}, (q-1)^{r-j+1})$$

Using (6.3) and taking the factor  $q$  inside, the second sum becomes

$$(6.4) \quad d \cdot \left( \sum_{j=0}^{r-1} s_j(\mathbf{d} - \mathbf{1}, (q-1)^{r-j+1}) \cdot q^{r-j} \mu_{r-1-j}(\mathcal{A}'') \right)$$

Thus, if we add (6.2) to (6.4) we obtain

$$(6.5) \quad d \cdot s_r(\mathbf{d} - \mathbf{1}, q-1) \mu_0(\mathcal{A}') \\ + d \cdot \left( \sum_{j=0}^{r-1} s_j(\mathbf{d} - \mathbf{1}, (q-1)^{r-j+1}) \cdot q^{r-j} (\mu_{r-j}(\mathcal{A}') + \mu_{r-1-j}(\mathcal{A}'')) \right)$$

Then,  $\mu_0(\mathcal{A}') = 1 = \mu_0(\mathcal{A})$  and by (2) of Proposition 2.1,

$$\mu_{r-j}(\mathcal{A}) = \mu_{r-j}(\mathcal{A}') + \mu_{r-1-j}(\mathcal{A}'')$$

Hence, (6.5) becomes

$$d \cdot \left( \sum_{j=0}^r s_j(\mathbf{d} - \mathbf{1}, (q-1)^{r-j+1}) q^{r-j} \mu_{r-j}(\mathcal{A}) \right) = p(\mathbf{d}, q, \mathcal{A})$$

this establishes the first functional equation.

For the second equation, we use the formulas of Greuel-Hamm [GH] and Giusti [Gi] for the Milnor number of a homogeneous ICIS. If it has

multidegree  $\mathbf{d} = (d_1, \dots, d_{n-r})$ . We write it in an equivalent form as in the remark following Theorem 8.10 in [D2]

$$(6.6) \quad \mu(\mathbf{d}) = (-1)^{r+1} + d \cdot \sum_{j=0}^r \sigma_{r-j}((-1)^n) s_j(\mathbf{d})$$

where again  $d = \prod_{i=1}^{n-r} d_i$  and  $\sigma_k((-1)^n) = \sigma_k(-1, \dots, -1)$  with  $n$  factors  $-1$ . Now, we may apply the  $\tau$  function in [D3] and write the sum in (6.6) as  $\tau(\mathbf{D})$  where  $\mathbf{D}$  is the  $r \times (n-r+1)$  matrix

$$\mathbf{D} = \begin{pmatrix} d_1 - 1 & d_2 - 1 & \dots & d_{n-r} - 1 & -1 \\ d_1 - 1 & d_2 - 1 & \dots & d_{n-r} - 1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_1 - 1 & d_2 - 1 & \dots & d_{n-r} - 1 & -1 \end{pmatrix}$$

Then, by the definition of  $s_r$  in [D3, §2],

$$(6.7) \quad \begin{aligned} \tau(\mathbf{D}) &= s_r(d_1 - 1, d_2 - 1, \dots, d_{n-r} - 1, -1) \\ &= s_r(\mathbf{d} - \mathbf{1}, -1) \end{aligned}$$

Then, we let  $d_{n-r+1} = q$ , and  $\mathbf{d}' = (d_1, \dots, d_{n-r}, d_{n-r+1})$ . Using (6.7)

$$(6.8) \quad \begin{aligned} \mu(\mathbf{d}) + \mu((\mathbf{d}, q)) &= (-1)^{r+1} + d \cdot s_r(\mathbf{d} - \mathbf{1}, -1) \\ &\quad + (-1)^{r-1+1} + d \cdot d_{n-r+1} \cdot s_{r-1}(\mathbf{d}' - \mathbf{1}, -1) \end{aligned}$$

Then, (6.8) equals

$$(6.9) \quad d \cdot (s_r(\mathbf{d} - \mathbf{1}, -1) + d_{n-r+1} \cdot s_{r-1}(\mathbf{d}' - \mathbf{1}, -1))$$

We may rewrite the second term of (6.9)

$$(6.10) \quad \begin{aligned} d_{n-r+1} \cdot s_{r-1}(\mathbf{d}' - \mathbf{1}, -1) &= (d_{n-r+1} - 1) \cdot s_{r-1}(\mathbf{d}' - \mathbf{1}, -1) \\ &\quad + s_{r-1}(\mathbf{d}' - \mathbf{1}, -1) \end{aligned}$$

Also, by the ‘‘Generalized Pascal Relation’’ for  $s_j$  (see [D3, §2])

$$(6.11) \quad s_r(\mathbf{d}' - \mathbf{1}, -1) = s_r(\mathbf{d} - \mathbf{1}, -1) + (d_{n-r+1} - 1) \cdot s_{r-1}(\mathbf{d}' - \mathbf{1}, -1)$$

Thus, we may apply (6.10) and (6.11) to rewrite (6.9) as

$$(6.12) \quad d \cdot (s_r(\mathbf{d}' - \mathbf{1}, -1) + s_{r-1}(\mathbf{d}' - \mathbf{1}, -1))$$

We may apply the expansion property of the  $s_j$  functions (again see [D3, §2]) to obtain

$$(6.13) \quad s_r(\mathbf{d}' - \mathbf{1}, -1) = s_r(\mathbf{d}' - \mathbf{1}) + (-1) s_{r-1}(\mathbf{d}' - \mathbf{1}, -1)$$

Thus, substituting (6.13) into (6.12), we obtain for (6.9)

$$\begin{aligned} d \cdot s_r(\mathbf{d}' - \mathbf{1}) &= d \cdot s_r(\mathbf{d} - \mathbf{1}, d_{n-r+1} - 1) \\ &= d \cdot s_r(\mathbf{d} - \mathbf{1}, q - 1) = p(\mathbf{d}, q, \{H\}) \end{aligned}$$

The last equation results from  $\mu_j(\{H\}) = 0$  for all  $j > 0$ .

This completes the proof of the theorem.

Corollary 2 is an immediate consequence of Theorem 1. For Corollary 3, we need only observe that the relative singular Milnor number

$$\mu(X_0, A_0) = \mu(X_0) + \mu(A_0)$$

is exactly the number of relative bounding cycles, which by (5.13) yields the formula for the singular Milnor number.

Also, for the  $k$ -th higher multiplicities we also have for a generic  $k$ -plane  $\Pi$

$$(6.10) \quad \mu(\Pi \cap X_0, \Pi \cap A_0) = \mu(\Pi \cap X_0) + \mu(\Pi \cap A_0)$$

while

$$(6.11) \quad \mu_k(X_0) = \mu(\Pi \cap X_0) \quad \text{and} \quad \mu_k(A_0) = \mu(\Pi \cap A_0)$$

Thus, combining (6.10) and (6.11) with the result for the singular Milnor fiber gives the result for higher multiplicities.

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