# Nonlinear Sections of Nonisolated Complete Intersections

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## Introduction

By a series of discoveries during the past thirty years, a distinguished list of researchers have provided us with a marvelous vista of singularity theory as it applies to isolated singularities, especially isolated complete intersection singularities. This began with Milnor's seminal monograph on isolated hypersurfaces singularities [Mi], which introduced as a principal tool in the study of isolated singularities the Milnor fibration of the singularity. The basic results were extended by Hamm [Ha] to isolated complete intersection singularities (ICIS). There has followed a succession of revelations concerning the topology, local geometry, and deformation theory of ICIS using: De Rham cohomology and Gauss-Manin connection, intersection pairing and monodromy, mixed Hodge structures and spectrum, structure of discriminants, equsingularity via multiplicities; and deformation theory. We refer to e.g. [Lo] and [**?**]vol 2]AVG where many of these results are presented.

If we were to seek a comparable view of the more complicated nonisolated singularities, then the results for ICIS provide a virtual "wish list"of types of results to be obtained. However, now the vista is considerably clouded, lacking many of the details so apparent for ICIS, although revealing general features via techniques involving stratification theory, resolution of singularities, etc.

A sample of the kinds of questions involving nonisolated singularities which we will consider involve e.g.: topology of complements of hyperplane arrangements, the topology of boundary singularities of complete intersections, critical points of functions $f_1^{\lambda_1} \cdots f_r^{\lambda_r}$ appearing in hypergeometric functions, minimum $\mathcal{A}_e$–codimenson for germs $f : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ in a given contact class, de Rham cohomology of highly singular spaces, higher multiplicities à la Teissier for nonisolated singularities together with Buchbaum–Rim multiplicities of modules; as well as properties of discriminants and bifurcation sets for various notions of equivalence. These examples share no obvious common feature except ultimately they concern properties of highly nonisolated singular spaces.

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However, there are two key obstacles to extending results for ICIS to such highly singular spaces. First, the Milnor fibration no longer has the connectivity properties possessed in the ICIS case. By the Theorem of Kato–Matsumoto [KM], for a hypersurface singularity $(X, 0)$ of dimension n with singular set $\mathrm{sing}(X)$, the connectivity of the Milnor fiber decreases by $\dim_{\mathbb{C}}(\mathrm{sing}(X))$, i.e. it is in general only $(n-1) - \dim_{\mathbb{C}}(\mathrm{sing}(X))$ connected. Second, unlike isolated singularities, nonisolated singularities generally do not have a versal deformation. Hence, we lack a relation between the topology of Milnor fibers and the structure of discriminants.

Because of the first complication, there is a range of cohomology groups to be determined to understand the topology of the Milnor fiber. By comparison with the state of affairs reported by Randell [Ra1] at the end of the 70's, considerable advances for low dimensional singular sets has been accomplished through the work led by Siersma [**?**]1-4]Si and the Dutch group of Pellikaan [Pe1, Pe2], Van Straten [VS1], de Jong [dJ], etc. along with coworkers Massey [Ms1, Ms2] [MSi], Tibar [Ti], Nemethi [Ne], etc. This work will be reported on by Siersma at this conference. However, for highly singular spaces this method becomes increasingly difficult to apply.

We describe an alternate approach, which has its origins in algebraic geometry beginning with Hilbert. It applies to large classes of nonisolated singularities which arise as nonlinear sections of fixed "model nonisolated singularities". We use a Thom-Mather type of group of equivalence, $\mathcal{K}_V$, to analyze the singularities of such sections. This allows us to overcome both obstacles.

Beginning with joint work with Mond on hypersurfaces, we obtain for nonlinear sections of nonisolated singularities, a singular analogue of the Milnor fibration. Using a result of Lê [Lê1], we show it retains the same connectivity properties as for Milnor fibers of an ICIS. A crucial ingredient for further analyzing the topology is the notion of freeness introduced by Saito [Sa] for divisors. If the model singularities are "free divisors and complete intersections", then the freeness first provides the algebraic condition needed to compute the singular Milnor number as the length of a determinantal module. In the hypersurface case, the module is the normal space for another equivalence $\mathcal{K}_H$. It agrees with the $\mathcal{K}_V$–normal space in the weighted homogeneous case, yielding a "$\mu \geq \tau$"result.

Second, in the complete intersection case, there is a generalization of the Lê–Greuel formula which computes the relative singular Milnor number of a divisor on an ICIS as the length of a determinantal module. When the divisor has an isolated singularity, we recover the Lê–Greuel formula as a special case. However, it applies more generally to include "arrangements of hypersurfaces" on complete intersections, singular Milnor fibers for "boundary singularities" (or equivalently complete intersection singularities on a divisor at infinity), projections of discriminants, etc. It further extends to general nonisolated complete intersections, where there is also an alternate approach, which computes the singular Milnor number of a nonisolated complete intersection as an alternating sum of singular Milnor numbers of unions of hypersurfaces.

These results allow us to introduce and compute higher multiplicities à la Teissier for nonisolated complete intersections. For hyperplane arrangements and their nonlinear generalizations, these multiplicities have surprising relations with other invariants of arrangements and with the topology of the complement.

To further explore the topology of nonlinear sections, we describe how methods using de Rham cohomology can be introduced. These include results of Alexandrov and Mond and coworkers which compute cohomology using the complex of logarithmic forms. Besides being used to compute the cohomology of the complement of a free divisor [CMN], and certain local cohomology [Av1, Av2], the logarithmic complex is used by Mond to construct the correct complex for computing the de Rham cohomology of the singular Milnor fiber and defining a Gauss–Manin connection for the singular Milnor fibration [Mo3].

Lastly, we turn in Part IV to consider why discriminants and bifurcation sets so frequently are free divisors. Except for results concerning discriminants for space curves and for functions on them by [VS2], [Go2], [MVS], results on freeness of discriminants can be summarized by the motto "Cohen–Macaulay of codim 1 + genericity of Morse–type singularities implies freeness".

We indicate how this specifically applies for various equivalences, as well as when it does not. When either condition fails this leads to the introduction of the notion of a "Cohen–Macaulay reduction" and a weaker *Free* Divisor structure*. However, the Free* Divisor structure still can be used to determine the topological structure as above.

One final point concerns the role that the structure of modules, especially determinantal modules, and (implicitly) Buchbaum–Rim multiplicities play in all of this work. This changes the emphasis from invariants associated to local rings to invariants of modules of logarithmic vector fields. This parallels the increasingly important role played by modules, their integral closures, reductions, and Buchbaum–Rim multiplicities in the work of Gaffney on Whitney equisingularity [Ga, Ga2], [GK], as well as its influence on the work of Kirby and Rees [KR] [Re], Kleiman–Thorup [KT], and Henry–Merle [HM] (as well as the many other references to be given in Gaffney's lectures). Because of this we raise several natural questions regarding the intrinsic nature of the results we describe and the form they might take as we move "beyond freeness"

# I Nonisolated Singularities as Nonlinear Sections

## 1. Singularities arising as Nonlinear Sections and $\mathcal{K}_V$–equivalence

We consider the approach to singularities which represents certain singularities and their deformations as sections of standard model singularities. For example, by the Hilbert–Burch theorem [Hi], [Bh], Cohen–Macaulay singularities of codimension 2 can be represented by the $n \times n$ minors of an $n \times (n+1)$ matrix of holomorphic germs. This was extended to their deformations by Schaps [Sh]. Buchsbaum–Eisenbud characterized codimension 3 Gorenstein singularities [BE] as the Pfaffians of principal minors of skew–symmetric matrices. A general criterion was given by Buchweitz [**?**]Chap 4, 5]Bu who defined a deformation theory for sections and showed that for certain singularities, their deformations can be represented as nonlinear sections of "very rigid singularities". Not all deformations of singularities can be so represented, as illustrated by the example of Pinkam [**?**]8.2]Pi of a surface singularity which is a cone on a rational curve of degree 4 in $\mathbb{P}^4$ and can be represented as the $2 \times 2$ minors of either a certain $2 \times 4$ matrix or a $3 \times 3$ symmetric matrix. Each representation gives rise to distinct components in the base of the versal deformation.

$$\begin{tikzcd} & \mathbb{C}^n, 0 \arrow[r, "f_0"] & \mathbb{C}^p, 0 \\ f_0^{-1}(V) \arrow[r, equal] & V_0, 0 \arrow[u] \arrow[r] & V, 0 \arrow[u] \end{tikzcd} \tag{1}$$

We shall be concerned instead with the properties of nonisolated singularities and their deformations obtained from $V$ by nonlinear sections $f_0$ and their unfoldings (this ignores the problem of missing (flat) deformations). Given $V, 0 \subset \mathbb{C}^p, 0$, we consider holomorphic germs $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ which we view as nonlinear sections of $V$ so that the singularity $V_0 = f_0^{-1}(V)$ is a pullback as given in (1). Here $f_0$ does not have to be a germ of an embedding.

EXAMPLE 1.1. Any germ $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ of finite singularity type, has a stable unfolding $F : \mathbb{C}^{n'}, 0 \to \mathbb{C}^{p'}, 0$. If $g_0 : \mathbb{C}^p, 0 \to \mathbb{C}^{p'}, 0$ denotes the inclusion, then the discriminant $D(f_0) = g_0^{-1}(D(F))$, with

$D(F)$ the discriminant of $F$. If $n < p$, then $D(F)$ is the image of $F$. How can the properties of $D(f_0)$ be deduced from those of $g_0$ and $D(F)$?

For properties of nonlinear sections $f_0$ of $V,0 \subset \mathbb{C}^p,0$, we follow the Thom–Mather approach by defining a group of equivalences $\mathcal{K}_V$ [D1] acting on sections $f_0$ and their unfoldings to capture the ambient equivalence of $V_0, 0 = f_0^{-1}(V), 0$. $\mathcal{K}_V$ is a subgroup of the contact group $\mathcal{K}$, introduced by Mather [M-II] (also see Tougeron [Tg]), and consists of germs of diffeomorphisms of $\mathbb{C}^{p+n},0$ of the form $\Psi(x,y) = (\Psi_1(x), \Psi_2(x,y))$.

$$\mathcal{K}_V \quad = \quad \{\Psi \in \mathcal{K} : \Psi(\mathbb{C}^n \times V) \subseteq \mathbb{C}^n \times V\}. \tag{2}$$

It acts on sections by the restriction of the action of $\mathcal{K}$: graph$(\Psi \cdot f_0) = \Psi(\text{graph}(f_0)))$.

We can extend $\mathcal{K}_V$ to the group of unfolding–equivalences $\mathcal{K}_{V,un}$ acting on unfoldings with any fixed number of unfolding parameters $u \in \mathbb{C}^q$. These groups (together with the associated unfolding groups) are "geometric subgroups of $\mathcal{A}$ or $\mathcal{K}$"and satisfy the basic theorems of singularity theory, especially the versality and finite determinacy theorems [D0]. In fact, it is shown in [D1] that such results also hold for $\mathcal{K}_V$–equivalence in the real case for smooth germs provided that $V$ is real analytic and coherent (in the sense of Malgrange [Mg]).

To define the associated extended tangent spaces (which are the deformation theoretic tangent spaces), we must introduce the module of "logarithmic vector fields". We let $\theta_p$ denote the module of germs of vector fields on $\mathbb{C}^p,0$. If $V,0 \subset \mathbb{C}^p,0$ is a germ of an analytic set, let $I(V)$ denote the ideal of germs vanishing on $V$. Then (following Saito [Sa]) we define

$$\text{Derlog}\,(V) = \{\zeta \in \theta_p : \zeta(I(V)) \subseteq I(V)\}.$$

This is the module of *logarithmic vector fields*, which are vector fields on $\mathbb{C}^p,0$ tangent to $V$. If Derlog$\,(V)$ is generated by $\zeta_0, \ldots, \zeta_r$, the extended $\mathcal{K}_V$ tangent space is computed [D1]

$$T\mathcal{K}_{V,e} \cdot f_0 \;=\; \mathcal{O}_{\mathbb{C}^n,0}\{\frac{\partial f_0}{\partial x_1}, \ldots, \frac{\partial f_0}{\partial x_n}, \zeta_0 \circ f_0, \ldots, \zeta_r \circ f_0\} \tag{3}$$

(the $R$–module generated by $\varphi_1, \ldots, \varphi_k$ is denoted by $R\{\varphi_1, \ldots, \varphi_k\}$, or just $R\{\varphi_i\}$ if $k$ is understood). The analogue of the deformation tangent space $T^1$ is the extended $\mathcal{K}_V$ normal space

$$N\mathcal{K}_{V,e} \cdot f_0 \quad = \quad \theta(f_0)/T\mathcal{K}_{V,e} \cdot f_0 \quad \simeq \quad \mathcal{O}^{(p)}_{\mathbb{C}^n,0}/T\mathcal{K}_{V,e} \cdot f_0$$

We give an important property of this equivalence (see [**?**]§1]D2.

EXAMPLE 1.2. *Invariance under suspension and projection* If $i : \mathbb{C}^p, 0 \hookrightarrow \mathbb{C}^{p+r}, 0$ denotes inclusion and $\pi : \mathbb{C}^{p+r}, 0 \to \mathbb{C}^p, 0$ projection, then as $\mathcal{O}_{\mathbb{C}^n,0}$–modules,

$$N\mathcal{K}_{V,e} \cdot f_0 \simeq N\mathcal{K}_{V\times\mathbb{C}^r,e} \cdot i \circ f_0 \quad \text{and} \quad N\mathcal{K}_{V\times\mathbb{C}^r,e} \cdot f \simeq N\mathcal{K}_{V,e} \cdot \pi \circ f_0$$

Moreover, suspension and projection preserve equivalence classes for $\mathcal{K}_V$ and $\mathcal{K}_{V\times\mathbb{C}^r}$. Hence, for investigating nonlinear sections we may replace $V$ by $V \times \mathbb{C}^q$ and retain both the topological and deformation theoretic properties of the germ $V_0$.

*Algebraic and Geometric Transversality*

Although any analytic germ $V_0, 0 \subset \mathbb{C}^n, 0$ is the zero set ( $= f_0^{-1}\{0\}$) for some analytic germ $f_0$, in general such a germ $f_0$ will not be transverse to 0 off $0 \in \mathbb{C}^n$. In order to ensure that properties of $V$ are passed on to $V_0$, we require that $f_0$ be transverse to $V$ outside 0. The notion of transversality we use depends upon the interpretation of "$T_yV$"at a singular point $y \in V$. If we use $T_yS_i$, where $S_i$ is the stratum of the canonical Whitney stratification of $V$, then we obtain "geometric transversality". However, for algebraic considerations, the appropriate version of transversality is more subtle, and invokes Derlog$(V)$.

For an $\mathcal{O}_{\mathbb{C}^p,0}$–submodule $M \subset \theta_p$ generated by $\{\zeta_1, \ldots, \zeta_r\}$, we let $\langle M\rangle_y$ be the subspace of $\mathbb{C}^p$ generated by $\{\zeta_{1(y)}, \ldots, \zeta_{r(y)}\}$. This is well–defined for $y$ in a neighborhood of 0. Then, we define the "logarithmic tangent spaces "$T_{log}V_{(y)} = \langle \text{Derlog}\,(V)\rangle_y$. Then, $f_0$ is algebraically transverse to $V$ at $x \in \mathbb{C}^n$ if

$$df_0(T_x\mathbb{C}^n) + T_{log}V_{(f_0(x))} = T_{(f_0(x))}\mathbb{C}^p$$

Then, just as for $\mathcal{K}$ equivalence, by [D1] there is a geometric characterization for $f_0$ having *finite* $\mathcal{K}_V$*–codimension* (i.e. $\dim_{\mathbb{C}}(N\mathcal{K}_{V,e} \cdot f_0) < \infty$); namely, $f_0$ has finite $\mathcal{K}_V$ codimension iff $f_0$ is algebraically transverse to $V$ off 0, i.e. at all $x$ in a punctured neighborhood of 0. We can analogously define algebraic (or geometric) transversality of germs of singular varieties, as well as algebraic (or geometric) general position. We always have

$$T_{log}V_{(y)} \;\subseteq\; T_yS_i \tag{4}$$

Hence,

algebraic transversality $\Longrightarrow$ geometric transversality.

However, (4) may be strict inclusion, and then the algebraic tangent spaces are tangent to a (possibly singular) foliation of a canonical statum. The strata $S_i$ for which (4) is equality at all points $y \in S_i$ are called *holonomic strata*, and the codimension of the complement of the set of holonomic strata is called the *holonomic codimension* and denoted $hn(V)$. If $n < hn(V)$, then the reverse implication in (4) holds.

*Low $\mathcal{K}_V$–codimension germs*

$f_0$ has $\mathcal{K}_{V,e}$–codimension 0 iff it is algebraically transverse to $V$ at 0. By the versality theorem, such an $f_0$ is already $\mathcal{K}_V$–versal, so any unfolding of $f_0$ is $\mathcal{K}_V$–trivial. If moreover $f_0$ is a germ of an embedding, then by the versality theorem, $V \simeq V_0 \times \mathbb{C}^{p-n}$.

For $\mathcal{K}_{V,e}$–codimension 1 we give a definition.

DEFINITION 1.3. *Given $V, 0 \subset \mathbb{C}^p, 0$ and an integer $n > 0$, then a germ $g : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ is a Morse–type singularity in dimension $n$ if $g$ has $\mathcal{K}_{V,e}$–codim $= 1$ and is $\mathcal{K}_V$–equivalent to a germ $f_0$, so that for a common choice of local coordinates, both $f_0$ and $V$ are weighted homogeneous. We furthermore say $V$ has a Morse–type singularity in dimension $n$ at $x$ if there is a germ $g : \mathbb{C}^n, 0 \to \mathbb{C}^p, x$ which is a Morse–type singularity (using $\mathcal{K}_{(V,x)}$–equivalence).*

Although $V$ is unspecified, such singularities can be precisely classified using $\mathcal{K}_V$–equivalence [**?**]Lemma 4.12]D7 and [**?**]Lemma 7.2]D8.

LEMMA 1.4. (Local Normal Form). *Let $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ be a Morse–type singularity for $V, 0 \subset \mathbb{C}^p, 0$. Then, up to $\mathcal{K}_V$–equivalence, we may assume $V, 0 = \mathbb{C}^r \times V', 0$ for $V', 0 \subset \mathbb{C}^{p'}, 0$ with $T_{log}V'_{(0)} = 0$, and with respect to coordinates for which $V', 0$ is weighted homogeneous, $f_0$ has the form*

$$f_0(x_1, \dots, x_n) = (0, \dots, 0, x_1, \dots, x_{p'-1}, \sum_{j=p'}^{n} x_j^2) \qquad (5)$$

*Remark.* The condition of weighted homogeneity in Definition 1.3 is not needed to obtain the normal form (5). However, it is essential so that $V$ will still be weighted homogeneous in (5) for questions involving freeness of discriminants in part IV.

The actual singularity obtained as a nonlinear section depends upon $V$ and can vary considerably, see e.g. the possible Morse–type singularities in dimension 2 for discriminants of stable multigerms [**?**]§4]D7. Nonetheless, we see in §3 that the topology (i.e. homotopy type) of a Morse–type singularity exactly matches ordinary Morse singularities.

*Relation between $\mathcal{A}$ and $\mathcal{K}_V$–equivalence*

To see the relevance of sections for other equivalences, we reconsider (1) except now allow $f_0$ to be a multigerm with $F$ its stable unfolding.

THEOREM 1.5. ([D2]). *There are the following relations between $\mathcal{A}$–equivalence for $f_0$ and $\mathcal{K}_{D(F)}$–equivalence for $g_0$:*

1. *$f_0$ has finite $\mathcal{A}$–codimension iff $g_0$ has finite $\mathcal{K}_V$–codimension;*

2. *if either is finite, then there is the isomorphism of $\mathcal{O}_{\mathbb{C}^p,0}$–modules*

$$N\mathcal{A}_e \cdot f_0 \quad \simeq \quad N\mathcal{K}_{V,e} \cdot g_0$$

*Let $g$ be an unfolding of $g_0$, with pullback of $F$ denoted by $f$.*

3. *If $g$ is a $\mathcal{K}_V$–trivial unfolding (resp. family) then $f$ is an $\mathcal{A}$–trivial unfolding (resp. family);*

4. *$g$ is $\mathcal{K}_V$–versal iff $f$ is $\mathcal{A}$–versal.*

*Remark.* Such a classification by sections has several extensions such as for $\mathcal{K}$–equivalence of unfoldings for [MM], $\mathcal{A}$–equivalence of unfoldings of hypersurface germs [**?**]§11]D4, etc.

The preceding theorem relates the $\mathcal{A}$ properties and classification of germs with those for $\mathcal{K}_V$–equivalence of sections. For a stable simple multigerm $F$, Morse–type singularities for sections of $V = D(F)$ provide the $\mathcal{A}_e$–codimension 1 multigerms in the contact class of $F$. The classification of $\mathcal{A}_e$–codimension 1 germs for simple $\mathcal{K}$ classes follows from Goryunov [Go1]; and for unfoldings of simple hypersurface germs and ICIS surface singularities, using local duality and results from Wahl, Looijenga, and this author (described in [**?**]§6]D7). These yield the $\mathcal{K}_{V,e}$–codimension 1 germs for discriminants of simple stable germs. Then, a construction using the product union [D7] gives the $\mathcal{K}_{V,e}$–codimension 1 germs for the discriminants of multigerms. In turn, this gives the $\mathcal{A}_e$–codimension 1 multigerms. Other approaches to their classification are given in Rieger [Ri] and by Wik Atique, Cooper, and Mond [ACM].

## 2. Role of Freeness for Divisors and Complete Intersections

To proceed further with properties of nonlinear sections, we need further information about $(V, 0)$. Rather than consider its local ring, we seek instead conditions on $\operatorname{Derlog}(V)$ which reveal the properties and topology of $V$, and of the singularities arising as nonlinear sections of $V$. In fact, Derlog$(V)$ is a Lie algebra; and Hauser and Müller [HMu] prove that the isomorphism class of Derlog$(V)$ in an appropriate sense uniquely determines $(V,0)$.

Among local rings, regular local rings are the simplest and correspond to smooth submanifolds. The simplest structure for Derlog$(V)$ occurs when $V,0 = \mathbb{C}^{p-1},0 \subset \mathbb{C}^p,0$ is a smooth hypersurface. Then, Derlog$(V)$ is a free module generated by $\{\frac{\partial}{\partial y_1},\dots,\frac{\partial}{\partial y_{p-1}}, y_p\frac{\partial}{\partial y_p}\}$. However, as discovered by Saito [Sa], and independently by Arnold [A2], this property does not characterize smooth hypersurfaces $(V,0)$. There are many important highly singular hypersurface singularities for which Derlog$(V)$ is free of rank $p$. It is still true that the freeness reveals much about their topology and that of nonlinear sections. This leads to Saito's definition [Sa].

DEFINITION 2.1. *A hypersurface $V,0 \subset \mathbb{C}^p,0$ is called a Free Divisor if* Derlog$(V)$ *is a free $\mathcal{O}_{\mathbb{C}^p,0}$–module. (necessarily of rank $p$)*

Initially three basic classes of free divisors were identified.

THEOREM 2.2. (Three "original" classes of free divisors ).

1. *Discriminants of versal unfoldings of isolated hypersurface and complete intersection singularities are free divisors (Saito* [Sa] *and Looijenga* [Lo]*);*

2. *Bifurcation sets of (versal unfoldings of) isolated hypersurface singularities are free divisors (Bruce* [Br] *and Terao* [To2]*);*

3. *Coxeter arrangements (union of reflecting hyperplanes for a Coxeter group) are free divisors (Terao* [To1]*).*

These examples illustrate how free divisors arise among fundamental objects, but (as we shall later see) this list only scratches the surface.

Saito also recognized for free divisors the important properties which follow for the complex of "logarithmic differential forms". Let $\Omega^k_{\mathbb{C}^p,0}$ denote the module of germs of holomorphic $k$–forms on $\mathbb{C}^p,0$. For a hypersurface germ $V,0 \subset \mathbb{C}^p,0$, with reduced defining equation $h$ we follow [Sa] and define the logarithmic $k$–forms

$$\Omega^k(\log V) \quad = \quad \{\omega \in \Omega^k_{\mathbb{C}^p\backslash V,0} : h\omega, hd\omega \in \Omega^k_{\mathbb{C}^p,0}\}$$

Let $\Omega^\bullet(\log V)$ denote the corresponding complex of logarithmic forms.

THEOREM 2.3. ([**?**]§1]Sa).

1. *The complex $\Omega^\bullet(logV)$ is an exterior algebra closed under exterior differentiation. Moreover, it is closed under interior product with, and Lie derivative by, vector fields in* Derlog $(V)$.

2. *$V, 0$ is a free divisor iff $\Omega^1(logV)$ is free of rank $p$; and then $\Omega^k(logV) = \bigwedge^k \Omega^1(logV)$ and is hence a free $\mathcal{O}_{\mathbb{C}^p,0}$–module.*

Furthermore, Saito gives an extremely useful criterion for freeness of a divisor.

THEOREM 2.4. (Saito's Criterion). *Let $\zeta_i \in$ Derlog $(V), i = 1, \dots, p$ with $\zeta_i = \sum_j a_{ij} \frac{\partial}{\partial y_j}$. If $\det(a_{ij})$ is a reduced defining equation for $V, 0$, then $V$ is a free divisor and the $\zeta_i$ generate* Derlog $(V)$.

This is the basis for many results identifying free divisors. The only other intrinsic characterization of freeness is given by Alexandrov [Av1], [Av2].

THEOREM 2.5. *Let $h$ be a reduced defining equation for a hypersurface germ $V, 0 \subset \mathbb{C}^p, 0$. Also let $J(h)$ denote the Jacobian ideal of $h$. Suppose Sing$(V)$ has codimension $1$ in $V, 0$, then each of the following is equivalent to $V$ being a free divisor:*

1. *Sing$(V)$ is a determinantal germ;*

2. *Sing$(V)$ is Cohen–Macaulay (for the structure given by $J(h)$).*

Using this criterion, Alexandrov gives an alternate proof that discriminants of versal unfoldings of ICIS are free divisors [Av2]

The parenthetical condition in 2) is crucial, as the intrinsic geometric structure of Sing$(V)$ does not determine whether $V$ is free

EXAMPLE 2.6. Consider the surface singularities in $\mathbb{C}^3$ defined by

$$f_1 = x^{10} + y^{10} + zx^6y^6 \quad \text{and} \quad f_2 = x^{10} + y^{10} + z(x^7y^5 + x^5y^7)$$

Both $f_i$ are equisingular deformations (with parameter $z$) of the plane curve singularity $f_0 = x^{10} + y^{10}$. Using different methods in [D11], it is shown that the first defines a free divisor while the second does not. However, each have the same singular set consisting of the $z$–axis, which is smooth and hence Cohen–Macaulay. Thus, the extra structure of $J(h)$ is required for Alexandrov's criterion. Moreover, both functions, viewed as deformations, are topologically equivalent to the trivial deformation of $f_0$. Thus, freeness is NOT a topological property of divisors. This contrasts with a conjecture of Terao that for central hyperplane arrangements (viewed as nonisolated singularities) the freeness is determined by the associated lattice.

The suspension of a free divisor $V \times \mathbb{C}^r, 0$ is easily seen to be free; however, a product of free divisors is not even a hypersurface. The product is naturally replaced by the "product union"of free divisors $V_i, 0 \subset \mathbb{C}^{p_i}, i = 1, 2$

$$V_1 \uplus V_2 \quad = \quad V_1 \times \mathbb{C}^{p_2} \cup \mathbb{C}^{p_1} \times V_2$$

(we can repeat the construction inductively). By [**?**]prop. 3.1]D4 the product union of free divisors is again a free divisor. For example, by Mather's multitransversality characterization of stability [M-V], the discriminant of a stable multigerm is the product union of the discriminants of the individual stable germs, and so free. Also, the "product"of hyperplane arrangements [**?**]Chap. 1]OT1 is really a product union.

Derlog $(H)$ *and Free Complete Intersections*

To extend the preceding to complete intersections, we encounter a basic problem. We would like $\{0\} \subset \mathbb{C}^p$ to be a free complete intersection; however Derlog $(\{0\})$ is generated by $\{y_i \frac{\partial}{\partial y_j}\}$ and is far from being free. We change our perspective to circumvent this problem. We recall from [**?**]§2]DM that we may always replace a hypersurface $V \subset \mathbb{C}^p$ by $V' = V \times \mathbb{C}$ and find a "good defining equation"$H$ for $V'$, which means there is an "Euler–like vector field"$e$ such that $e(H) = H$. This does not require that $V$ be weighted homogeneous (although if it is and of nonzero weight then the usual Euler vector field suffices). As this does not alter properties of nonlinear sections, we may suppose $V$ already has this property. Then, we introduce the module of vector fields annihilating $H$.

$$\text{Derlog}\,(H) \quad = \quad \{\zeta \in \theta_p : \zeta(H) = 0\}. \tag{6}$$

Then, for example by [**?**]lemma 3.3]DM (or equivalently see [Av1])

$$\text{Derlog}\,(V) \quad = \quad \text{Derlog}\,(H) \,\oplus\, \mathcal{O}_{\mathbb{C}^p,0}\{e\}$$

Hence, if $V$ is a free divisor with good defining equation, then Derlog $(H)$ is a free $\mathcal{O}_{\mathbb{C}^p,0}$–module of rank $p-1$ and conversely.

For the corresponding notion of freeness for a complete intersection $V, 0 \subset \mathbb{C}^p, 0$ defined by $H : \mathbb{C}^p, 0 \to \mathbb{C}^k, 0$, we define Derlog $(H)$ as in (6).

DEFINITION 2.7. *A complete intersection $V, 0 \subset \mathbb{C}^p, 0$ defined by $H : \mathbb{C}^p, 0 \to \mathbb{C}^k, 0$ is a (H–) free complete intersection if* Derlog $(H)$ *is a free $\mathcal{O}_{\mathbb{C}^p,0}$–module (necessarily of rank $p-k$).*

EXAMPLE 2.8. The most basic example of a free complete intersection is given by the product of free divisors, see [**?**]§5]D8 (although there are other examples given there). Thus, $\{0\} \subset \mathbb{C}^p$ is a free complete intersection (with Derlog $(H) = 0$).

While free divisors (and free complete intersections) can be thought of as rigid universal objects, a much larger and richer class of divisors arise as nonlinear sections of free divisors by maps algebraically transverse off 0.

DEFINITION 2.9. *A hypersurface $V', 0 \subset \mathbb{C}^n, 0$ is an almost free divisor (AFD) (based on $V$) if $V' = f_0^{-1}(V)$ where $V, 0 \subset \mathbb{C}^p, 0$ is a free divisor and $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ is algebraically transverse to $V$ off 0. Similarly $V', 0$ is an almost free complete intersection (AFCI) if instead $V, 0$ is a free complete intersection.*

Each class of free divisors (or free complete intersections) yields a much larger corresponding class of almost free divisors (or almost free complete intersections).

EXAMPLE 2.10. *Examples of AFD and AFCI* i) By (2.8) any ICIS is an AFCI. ii) By Theorems 2.2 and 1.5, the discriminant of any finitely $\mathcal{A}$–determined germ which defines an ICIS is an AFD. iii) A central arrangement of hyperplanes in general position off 0 is an AFD which is the pullback of a Boolean arrangement, consisting of coordinate hyperplanes.

EXAMPLE 2.11. *Key properties of almost free divisors and complete intersections* (see [**?**]§3 and §7]D4 and [**?**]§5]D8).

1. The pullback of an almost free divisor or complete intersection by a finite map germ algebraically transverse off 0 is again an almost free divisor or complete intersection.

2. If $(V_i, 0)$ are almost free divisors "transverse"off 0, i.e. in algebraic general position off 0 then

   i) the "transverse union"$(\cup V_i, 0)$ is again an almost free divisor and;

   ii) the "transverse intersection"$(\cap V_i, 0)$ is an almost free complete intersection.

EXAMPLE 2.12. *Examples resulting from properties* i) The transverse union of isolated hypersurface singularities is not isolated; however, it remains an almost free divisor. ii) (boundary singularities) If $V, 0 \subset$

$\mathbb{C}^p, 0$ is a free divisor and $X, 0 \subset \mathbb{C}^p, 0$ is an ICIS (algebraically) transverse to $V$ off 0, then $(V \cap X, 0)$ is an AFCI. iii) $(V, 0)$, the union of 4 lines through 0 in $\mathbb{C}^3$, is an ICIS; but it is also an AFCI obtained as the pullback by a general 3–dimensional linear section of the free complete intersection obtained as the product of the Boolean arrangement $\{(x, y) : xy = 0\}$ with itself. In this last example, the different ways of viewing $(V, 0)$ are reflected in the different ways of deforming it and obtaining vanishing topology in part II.

# II Topology of Nonlinear Sections

## 3. Topology of Singular Milnor Fibers

For a nonlinear section of a nonisolated complete intersection we introduce a (topological) stabilization. Via this stabilization, we define a "singular Milnor fibration" of $f_0$ so that the "singular Milnor fiber" will have the correct connectivity properties. In fact, the "singular Milnor fiber" will be homotopy equivalent to a bouquet of spheres of correct dimension. The number of such spheres defines a "singular Milnor number". We then describe how to algebraically compute the singular Milnor number as the length of a determinantal module, which turns out to be the $\mathcal{K}_{H,e}$–normal space of $f_0$.

*Stabilization of Nonlinear sections of Complete Intersections*

Suppose $V, 0 \subset \mathbb{C}^p, 0$ is a nonisolated complete intersection of codimension $k$. Let $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ be a nonlinear section of $V, 0$ which is geometrically transverse to $V$ off 0. By a *(topological) stabilization of* $f_0$ we mean a holomorphic family of maps $f_t : U \to \mathbb{C}^p$ (for $U$ a neighborhood of 0), such that: $f_t$ is geometrically transverse to $V$ for $t \neq 0$ (i.e. transverse to the canonical Whitney stratification of $V$); and for $t = 0$ it is a representative of the germ $f_0$ which is transverse to $V$ on $U \backslash \{0\}$. In the case $n < hn(V)$ we can even ensure the stabilization is algebraically transverse to $V$.

Then, we can apply a theorem of Lê [Lê1] which extends Milnor's theorem to a function germ $f$ defined on a (possibly nonisolated) $n+1$–dimensional complete intersection $\mathcal{X}, 0 \subset \mathbb{C}^m, 0$, and which has an isolated singularity in an appropriate sense. Lê proves $f_0$ has a Milnor fibration with fibers which are singular but still $n-1$ connected and homotopy equivalent to a bouquet of real $n$–spheres. Then, Lê's theorem can be appropriately applied to a projection on the stabilization and combined with [LêT2] and standard type stratification arguments.

*Figure 1.* Singular Milnor fibers for: a) hyper/ section of a discriminant and b) an AFCI obtained as the intersection of a braid arrangement and a quadric

Letting $B_\epsilon$ denote a ball about 0 in $\mathbb{C}^n$ of radius $\epsilon > 0$. We obtain for nonlinear sections of hypersurfaces from joint work with Mond [**?**]Thm 4.6]DM (and for complete intersections [**?**]Lemma 7.8]D4, or [**?**]Thm 3]D5).

THEOREM 3.1. *Let $f_t$ as above be a stabilization of $f_0$ as a non-linear section of the complete intersection $V, 0 \subset \mathbb{C}^p, 0$ of dimension $k$. For $t$ and $\epsilon > 0$ sufficiently small, $f_t^{-1}(V) \cap B_\epsilon$ is independent of the stabilization $f_t$ and is homotopy equivalent to a bouquet of real $(n-k)$–spheres.*

Then, $f_t^{-1}(V) \cap B_\epsilon$ is called the *singular Milnor fiber* of $f_0$ (or of $V_0$). The number of such spheres is called the *singular Milnor number* and denoted by $\mu_V(f_0)$ (or $\mu(V_0)$ if $f_0$ is understood). The spheres themselves are the *singular vanishing cycles.*

EXAMPLE 3.2. For the examples in (2.12): i) the singular Milnor fiber is the union of the Milnor fibers of the hypersurfaces, chosen so they are in general position; ii) (boundary singularities) the singular Milnor fiber of $V \cap X, 0$ is the intersection of the boundary singularity $V$ with the usual Milnor fiber of $X$ chosen so it is transverse to $V$, figure 1 b); iii) for the AFCI of 4 lines in $\mathbb{C}^3$, the singular Milnor fiber is a union of 4 skew lines $\ell_i$ such that they intersect in cyclic order $\ell_1$ and $\ell_2$, $\ell_2$ and $\ell_3$ etc., forming a single singular 1-cycle.

*Remark.* This extends to images of finite map germs $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^{n+1}, 0$, see Mond [Mo1] and [Mo2]

*Formula for the Singular Milnor number of an Almost Free Divisor*

We next give the algebraic formula for the singular Milnor number of an almost free divisor based on the free divisor $V$ with a good defining equation $H$. This formula is the analogue of Milnor's original formula.

To do so we introduce another equivalence, $\mathcal{K}_H$–equivalence, which is an analog of $\mathcal{K}_V$–equivalence for which $\Psi$ in (2) preserves instead the level sets of the defining equation $H$. The extended tangent space

$T\mathcal{K}_{H,e} \cdot f_0$ is computed by replacing in (3) the generators of Derlog $(V)$ by Derlog $(H)$, i.e. we remove the Euler–like vector field $e$. (see [**?**]§3]DM).

As $V$ is free, we may choose generators $\{\zeta_0, \zeta_1, \dots, \zeta_{p-1}\}$ for Derlog $(V)$ such that $\zeta_0 = e$ and $\zeta_i, i > 0$. generate Derlog $(H)$.

We let $\nu_V(f_0)$ denote an algebraic codimension ($= \mathcal{K}_{H,e}-$codim $(f_0)$) which is defined by

$$\nu_V(f_0) = dim_{\mathbb{C}}\ \mathcal{O}_{\mathbb{C}^n,0}\{\frac{\partial}{\partial y_i}\}/\mathcal{O}_{\mathbb{C}^n,0}\{\frac{\partial f_0}{\partial x_1}, \dots, \frac{\partial f_0}{\partial x_n}, \zeta_1 \circ f_0, \dots, \zeta_{p-1} \circ f_0\} \quad (7)$$

Just as for Derlog $(V)$, we can define

$$T_{log}(H)_{(y)} \ = < \text{Derlog}\,(H) >_{(y)}$$

Equality holds $T_{log}(H)_{(y)} = T_{log}(V)_{(y)}$ if there is an Euler–like vector field defined near $y$ which vanishes at $y$, e.g. if $V$ is locally weighted homogeneous at $y$ for some choice of local coordinates. Then, we define the $H$–holonomic codimension $h(V)$ as we did the holonomic codimension $hn(V)$, except using the condition $T_{log}(H)_{(y)} = T_y S_i$, i.e. $h(V)$ is the codimension of the largest stratum which is not $H$–holonomic.

Then, provided we remain below $h(V)$ we can compute the singular Milnor number using (7)

THEOREM 3.3. ([**?**]Thm 5]DM). *Suppose that $V, 0$ is a free divisor with $n < h(V)$, and that $f_0$ is a germ of an embedding algebraically transverse to $V$ off 0 so that $\nu_V(f_0) < \infty$. Then, the singular Milnor number $\mu_V(f_0) = \nu_V(f_0)$.*

*Remark.* Originally in [DM], the result was stated for embeddings $f_0$; however, by the graph trick and invariance under suspension, we can apply the theorem to any $f_0$ algebraically tranverse to $V$ off 0 by reducing to an embedding, see e.g. the discussion in [**?**]Pt I]D4.

*$\mu \geq \tau$ results :* With the notation of (1.1), suppose $f_0$ is a finite $\mathcal{A}$–codimension germ with stable unfolding $F$. Let $\mu = \mu_{D(F)}(g_0)$ and $\tau = \mathcal{K}_{V,e} -$ codim $(f_0)$. Then, as a corollary of Theorem 3.3 (see [DM]), provided $(n, p)$ is in the "nice dimensions" in the sense of Mather [M-VI], $\mu \geq \tau$ with equality if $f_0$ is weighted homogeneous. This is the analogue of the $\mu = \tau$ results obtained by Greuel, Wahl, and Looijenga–Steenbrink see[**?**]Chap. 8]Lo. However, it can fail if $n \geq h(V)$ (see [**?**]Thm.6]DM and [**?**]§4]D9). We further consider other $\mu = \tau$ results and their relation to freeness of discriminants in Part IV.

EXAMPLE 3.4. i) In the case of isolated hypersurface singularities, we obtain the usual Milnor fibers and the usual formula for the Milnor numbers.

ii) If $g : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ is a Morse type singularity for $V$, then the weighted homogeneity implies $\mathcal{K}_{H,e}\text{codim}\,(g) = \mathcal{K}_{V,e}\text{codim}\,(g) = 1$. Hence, provided $n < h(V)$, the theorem implies that for Morse–type singularities, there is exactly one singular vanishing cycle in their singular Milnor fibers, as for usual Morse singularities.

iii) For the versal unfolding $F$ of any simple hypersurface singularity, $D(F)$ is weighted homogeneous. A linear section of $D(F)$ defined by the vanishing of unfolding parameter of lowest weight is a Morse–type singularity . A perturbation of the section moves off the origin and creates exactly one vanishing cycle as is usual/tured for the swallowtail singularity figure 1, also see [**?**]§4]D4).

Suppose $V, 0 \subset \mathbb{C}^p, 0$ is an almost free divisor based on $V', 0 \subset \mathbb{C}^{p'}, 0$ via $g_0 : \mathbb{C}^p, 0 \to \mathbb{C}^{p'}, 0$, and that $\nu_{V'}(g_0) < \infty$. Derlog $(V)$ is generally not free and Theorem 3.3 does not apply (and, in fact, fails badly) for sections of $V$. However by the behavior of almost free divisors under pullback by finite map germs, we can still compute the singular Milnor number of a section of $V$ (see [**?**]Cor. 4.2]D3).

COROLLARY 3.5. *Suppose that $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ is a finite map germ with $n < p$, $h(V')$, which satisfies $\nu_{V'}(g_0 \circ f_0) < \infty$. The singular Milnor number for $f_0$ (which equals the number of $(n-1)$–spheres in $f_t^{-1}(V) \cap B_\epsilon$ for a stabilization $f_t$) is given by $\nu_V(g_0 \circ f_0)$.*

## 4. An Extension of the Lê-Greuel Formula

*Relative Singular Milnor Number for a Divisor on a Complete Intersection*

The Milnor number of an ICIS is computed inductively via the Lê–Greuel formula. Suppose $X, 0 \subset \mathbb{C}^n, 0$ is a positive $n - m$–dimensional ICIS defined by $f_2 = (f_{21}, \dots, f_{2m})$, and that $f_1 : \mathbb{C}^n, 0 \to \mathbb{C}, 0$ has an isolated singularity restricted to $X$. Then, $f = (f_1, f_2)$ defines the ICIS $X_0 = X \cap f_0^{-1}(0)$.

THEOREM 4.1. (Lê–Greuel [LGr]). *In the preceding situation*

$$\mu(f) + \mu(f_2) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0}/((f_{21}, \dots, f_{2m}) + J(f)) \tag{8}$$

*$J(f)$ denotes the ideal generated by the $(m+1) \times (m+1)$ minors of $df$.*

Suppose $X_y$ denotes a Milnor fiber of $f_2$ over $y$ such that the intersection $X_{(0,y)} = X_y \cap f_1^{-1}(0)$ is transverse giving the Milnor fiber of $f$. Then, the pair $(X_y, X_{(0,y)})$ has homology only in dimension $n-m$ and the LHS of (8) is the "relative Milnor number"$= \dim_{\mathbb{C}} H^{n-m}(X_y, X_{(0,y)})$. Thus, the RHS of (8) can be alternately viewed as providing an algebraic formula for this relative Milnor number.

We extend this result to the relative case of an almost free divisor $V_0$ transversely intersecting an almost free complete intersection $X$ off 0. We begin with the case where $X$ is again an ICIS defined by $f_2$ as in Theorem 4.1. Now we suppose $f_1 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ is algebraically transverse to the free divisor $V$ off 0. Let $\{\zeta_1 \dots, \zeta_{p-1}\}$ be generators for Derlog $(H)$, with $H$ a good defining equation for $V$.

If $f_{1t}$ is a stabilization of $f_1$ and $V_t = f_{1t}^{-1}(V)$ is transverse to $X_y = f_2^{-1}(y) \cap B_\epsilon$ on $B_\epsilon$, then $X_{(t,y)} = V_t \cap X_y$ is the singular Milnor fiber of $V_0 \cap X$. Then, $((X_y, X_{(t,y)})$ is $n-m-1$–connected and the relative singular Milnor number $\mu_{(X,X\cap V_0)}(f) = \dim_{\mathbb{C}} H^{n-m}(X_y, X_{(t,y)})$ is computed via the following module version of the Lê–Greuel formula.

THEOREM 4.2. ([**?**]Cor. 9.6]D4). *For the relative case of an almost free divisor $V_0$ and ICIS $X$, as above, the relative singular Milnor number is given by*

$$\mu_{(X,X\cap V_0)}(f) = \dim_{\mathbb{C}} \mathcal{O}_{X,0}^{(p+m)} / \mathcal{O}_{X,0}(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}, \zeta_1 \circ f_1, \dots, \zeta_{p-1} \circ f_1)$$

We note several consequences. First, $(X, 0)$ is a complete intersection (and hence Cohen–Macaulay) of positive dimension $n - m$. The RHS in Theorem 4.2 is the quotient of a free $\mathcal{O}_{X,0}$–module of rank $p + m$, on $n + p - 1 = (n - m) + (p + m) - 1$ generators. As the quotient has finite length, by results of Eagon–Northcott the quotient module on the RHS in Theorem 4.2 is Cohen–Macaulay. In particular, its length is also its Buchsbaum–Rim multiplicity. Hence, by results of Buchsbaum–Rim [BRm], the length equals the length of the quotient algebra of $\mathcal{O}_{X,0}$ by the $(n+p-1)\times(n+p-1)$ minors of the matrix formed from the generators of the quotient. In particular, in the case $V = \{0\}$, there are no $\zeta_i$, and we obtain exactly the Lê– Greuel formula for the ICIS case.

Second, we may apply this result to the case of a fixed free divisor $V, 0 \subset \mathbb{C}^n, 0$ which we view as a "boundary"and consider "boundary singularities", e.g. [A3], [Ly], etc. (or equivalently view $V$ as a divisor at infinity and consider locally singularities at infinity, e.g. [SiT]). Then, $f_1 = id_{\mathbb{C}^n}$. Let $X_y$ be the Milnor fiber of an ICIS $g$ transverse to $V$. By projecting off the first $m$ generators, Theorem 4.2 takes the form.

COROLLARY 4.3. *For a ICIS $X, 0$ defined by $g : \mathbb{C}^n, 0 \to \mathbb{C}^m, 0$ for $n > m$, the boundary singularity for the free divisor boundary $V$ has relative singular Milnor number*

$$\dim_{\mathbb{C}} H^{n-m}(X_y, X_y \cap V) \;=\; \dim_{\mathbb{C}} \mathcal{O}_{X,0}^{(m)} / \mathcal{O}_{X,0}\{\zeta_1(g), \dots, \zeta_{n-1}(g)\}$$

In particular, for an isolated hypersurface singularity $X$, we recover a formula which is similar, although not identical, to one of Bruce–Roberts [**?**]Prop. 6.4]BR, except they were computing the number of critical points as opposed to the singular Milnor number. As both invariants are computed by counting critical points, these numbers should agree, despite the slight difference in the formulas.

If we consider the more general situation where now we replace the ICIS $X$ by a general AFCI based on a product of free divisors $V' \subset \mathbb{C}^{p_2}$, then we still obtain a formula given by the length of a determinantal module as in (4.2). However, it computes instead the relative Euler characteristic $\chi(X_y, X_{(t,y)})$ where $X_y$ is the usual smooth Milnor fiber of the germ $f_2$ (which is no longer a bouquet of $n-m$–spheres) and $X_{(t,y)}$ is the intersection of the singular Milnor fiber of $f_1$ with $X_y$. We denote this by $\tilde{\chi}(X, X \cap V_0)$. We let $V'$ be defined by $h_2 = (h_{22}, \ldots, h_{2m})$ as a product of free divisors and $V, 0$ by $h_1$, with each $h_{2i}$ and $h_1$ good defining equations for the appropriate free divisor. Let $h = (h_1, h_2)$.

THEOREM 4.4. ([**?**]Thm. 9.4]D4). *In the preceding situation,*

$$\tilde{\chi}(X, X \cap V_0) = (-1)^{\ell} \dim_{\mathbb{C}} \mathcal{O}_{X,0}^{(p+m)} / \mathcal{O}_{X,0}\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}, \zeta_1 \circ f, \ldots, \zeta_{n-m} \circ f\right)$$

*where* $\ell = n - m + 1$ *and* $\zeta_1, \ldots, \zeta_{n-m}$ *are generators for* Derlog $(h)$.

To relate this relative Euler characteristic with the relative singular Milnor number, we consider an alternate approach when $V$ is given as the transverse intersection off 0 of almost free divisors $V_i, i = 1, \ldots, m$. The transverse union of a subset of the $V_i$ is still an almost free divisor, so the singular Milnor number can be computed by Theorem 3.3. Then, the singular Milnor number of the intersection $\cap_i V_i$ can be expressed as an alternating sum of singular Milnor numbers of various unions of subsets of the $V_i$, see [**?**]§8, Thm. 2]D4. For example, this can be applied to give an alternate derivation of the formula of Guisti [Gi], Greuel–Hamm [GH], and Randell [Ra2] for the Milnor number of a weighted homogeneous ICIS.

In the simplest case the formula takes the form

$$\mu(V_1 \cap V_2) = \mu(V_1 \cup V_2) - \mu(V_1) - \mu(V_2) \tag{9}$$

For example, consider the complete intersection of two free divisors which are suspensions of cusp singularities as shown in figure 2. The singular Milnor fiber of both the intersection and union as shown have a single vanishing cycle (and each $\mu(V_i) = 0$). By contrast, Theorem 4.4 gives a relative Euler characteristic $= 9$ (using the smooth Milnor fiber). Hence, it not only counts the vanishing cycle but also adds contributions of 2 for each singularity in the singular Milnor fiber.

*Figure 2.* Intersection/union of free divisors, their singular Milnor fibers, with the single vanishing cycles

REMARK 4.5. [Proof of the relative formulas] As a main step in proving the Theorems 4.3 and 4.7, we apply a generalization of the lemma of Siersma used in [DM]. In [Si] Siersma shows that the standard Morse theory type argument used by Looijenga in [**?**]Chap. 5]Lo, and in its original form due to Lê [Lê2] [Lê3], can also be extended to nonisolated singularities defined by germs $g : \mathbb{C}^{n+1}, 0 \to \mathbb{C}, 0$. For the relative case, we establish [**?**]§9]D4, [**?**]§4]D6 an analogue for nonisolated singularities defined on complete intersections. Note these methods compute the topology of singular spaces without using stratified Morse theory of [GMc].

### *Buchbaum–Rim Multiplicity of a Determinantal module on a Complete Intersection*

The formulas for the singular Milnor number or relative singular Milnor number given in Theorems 3.3, 4.2, 4.4 and 4.3 are given by the length of a determinantal module (on a complete intersection). This is the Buchbaum–Rim multiplicity of the module. We next give a formula for this Buchbaum–Rim multiplicity in the (semi-) weighted homogeneous case in terms of the weights.

We suppose that $X, 0 \subset \mathbb{C}^p, 0$ is a weighted homogeneous complete intersection defined by $f = (f_1, \dots, f_{p-n})$ where $\mathrm{wt}\,(y_i) = a_i$, $\mathrm{wt}\,(f_i) = b_i$. Let $F_1, \dots, F_{n+k-1} \in (\mathcal{O}_{\mathbb{C}^p,0})^k$ be weighted homogeneous of degrees $d_i$. This means we assign weight $c_j$ to $\epsilon_j = (0, \dots, 1, 0, \dots, 0)$, with 1 in the $j$–th position. Then, $F_i = (F_{i1}, \dots, F_{ik})$ with each $F_{ij}$ weighted homogeneous of weight $d_i + c_j$ ($d_i$ and $c_j$ are not uniquely determined by $F_{ij}$). We let $\mathbf{a} = (a_1, \dots, a_p)$, $\mathbf{b} = (b_1, \dots, b_{p-n})$, and similarly for $\mathbf{d}$ and $\mathbf{c}$. Then, let

$$
\begin{aligned}
M &= (\mathcal{O}_{X,0})^k / (\mathcal{O}_{X,0}\{F_1, \dots, F_{n+k-1}\}) \\
R &= \mathcal{O}_{X,0} / I_k\,(F_1, \dots, F_{n+k-1})
\end{aligned}
$$

where $I_k(F_1, \dots, F_{n+k-1})$ denotes the ideal generated by the $k \times k$ minors of the $k \times (n+k-1)$ matrix $(F_{ij})$.

There is considerable redundancy in the sets of weights $\mathrm{wt}\,(F_{ij})$. There is a smaller $n \times k$ degree matrix $\mathbf{D}$ whose $ij$-th entry is $d_{i+j-1}+c_j$. Then, we define a universal function $\tau$ for all matrices [**?**]§2]D3; and using properties of determinantal modules due to Macaulay [Mc] and Northcott [N], we give a formula for the dimensions of $M$ and $R$ in terms of the weights via $\tau(\mathbf{D})$. Moreover, $\tau(\mathbf{D})$ can be expressed using $\sigma_j(\mathbf{d})$, the $j$-th elementary symmetric function in $d_1, \dots, d_{p+k-1}$, and $s_j(\mathbf{c})$ which is the sum of all monomials of degree $j$ in $c_1, \dots, c_k$.

THEOREM 4.6. *Both* $\dim_{\mathbb{C}} M$ *and* $\dim_{\mathbb{C}} R$ *are independent of both* $\{F_i\}$ *and* $f$ *and hence depend only on the weights* $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$*, provided the dimensions are finite. Moreover, in this case, these are the Buchsbaum–Rim multiplicity and are given by*

$$\dim_{\mathbb{C}} M \;=\; \dim_{\mathbb{C}} R \;=\; \frac{b}{a} \cdot \tau(\mathbf{D}) \;=\; \frac{b}{a} \cdot \sum_{j=0}^{p} s_{p-j}(\mathbf{c}) \cdot \sigma_j(\mathbf{d}) \qquad (10)$$

*where* $a = \prod a_i$ *and* $b = \prod b_i$.

*Remark.* This theorem was proven in [D3] for the case of $X = \mathbb{C}^n$. A fairly simple modification of that proof gives this more general result. The theorem extends to the semi–weighted homogeneous case, where for the initial parts of the $f_i$ and $F_j$, $M$ and $R$ are still finite dimensional.

EXAMPLE 4.7. The natural properties of the functions $\sigma_j$ and $s_j$ (see [**?**]§2]D3) suggest that formulas as above expressed in terms of them should have analogous forms in the nonweighted homogeneous case where the $\sigma_j$ and $s_j$ are replaced by appropriate invariants. For example, in the homogeneous case with $X = \mathbb{C}^p$ so $n = p$, there are no $f_i$, and all $a_i = 0$, $c_i = 0$, so that all $F_i$ are homogeneous with all components of degree $d_i$. Then, the Buchsbaum–Rim multiplicity is given by $\sigma_n(d_1, \dots, d_{n+k-1})$. We next see these have interpretations as "higher multiplicities".

## 5. Higher Multiplicities

Through singular Milnor fibers and numbers for nonlinear sections of complete intersections, we can introduce higher multiplicities á la Teissier for nonisolated complete intersections. Teissier [Te] defined for isolated hypersurface singularities a series of higher multiplicities, the $\mu^*$– sequence $\mu^* = (\mu_0, \dots, \mu_n)$. Given $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}, 0$, if $\Pi$ is a generic $k$–dimensional subspace in $\mathbb{C}^n$ then $f_0|\Pi$ has an isolated singularity and Teissier defines $\mu_k(f_0) = \mu(f_0|\Pi)$, where $\mu$ denotes the usual

Milnor number. Lê–Teissier [LêT] [Te3] further considered for general $(V, 0) \subset \mathbb{C}^n, 0$, the "k-th vanishing Euler characteristics" $\chi(\pi^{-1}(z) \cap V \cap B_\epsilon)$, where $\pi : V, 0 \to \mathbb{C}^k, 0$ is the restriction of a generic linear projection, $z$ is sufficently general, and $\|z\|$ and $\epsilon > 0$ are sufficiently small. They use the polar multiplicities combined with these vanishing Euler characteristics relative to strata of a Whitney stratification to compute topological invariants of nonisolated singularities.

For nonisolated complete intersections $V, 0 \subset \mathbb{C}^p, 0$, we can define higher multiplicities using the analogue of Teissier's definition for the hypersurface case. A Zariski open subset of $k$-dimensional subspaces $\Pi \subset \mathbb{C}^p$ are geometrically transverse to $V$ off 0 (which implies algebraic transversality if $k < h(V)$). We view the inclusion $i : \Pi \to \mathbb{C}^p$ as a section of $V$ and define $\mu_k(V) = \mu_V(i)$, the singular Milnor number of the section $i$ (if $V$ is itself a nonlinear section, $\mu_p(V)$ is its singular Milnor number). Then, $\mu_k(V)$ counts the number of singular vanishing cycles for a perturbation $i_t$ of the section $i$. If $k$ is one less than the codimension of the canonical Whitney stratum of $V$ containing 0, then the singular Milnor fiber of the section is the "complex link" of $V$ as defined by Goresky-MacPherson [GMc].

In the special case where $V$ is an almost free divisor based on $V', 0 \subset \mathbb{C}^{p'}, 0$ via $g_0$, Corollary 3.5 computes the higher multiplicities.

PROPOSITION 5.1. *Suppose that* $(V, 0)$ *is an almost free divisor based on* $(V', 0)$ *via* $g_0$. *Let* $i : \mathbb{C}^k, 0 \to \mathbb{C}^p, 0$ *be a linear section where* $k < p, h(V')$. *If* $\nu_{V'}(g_0 \circ i)$ *is finite and minimum among all nearby linear embeddings, then*

$$\mu_k(V) = \mu_{V'}(g_0 \circ i) = \nu_{V'}(g_0 \circ i).$$

This proposition and Theorem 4.3 allow us to compute higher multiplicities for weighted homogeneous free divisors $V, 0 \subset \mathbb{C}^p$. Let $\mathrm{wt}\,(y_i) = a_i$, with $a_1 \leq a_2 \leq \ldots \leq a_p$. Let $H$ be the weighted homogeneous defining equation, and suppose weighted homogeneous generators $\zeta_i$ for $\mathrm{Derlog}\,(H)$ have $\mathrm{wt}\,(\zeta_i) = d_i$.

PROPOSITION 5.2. *Suppose that the* $k$*–dimensional section of* $(V, 0)$ *defined by* $y_1 = \ldots = y_{p-k} = 0$ *is algebraically transverse to* $V$ *off* 0, *then*

$$\mu_k(V) = \frac{1}{a''} \sum_{j=0}^{k} s_{k-j}(\mathbf{a}') \cdot \sigma_j(\mathbf{d}) \tag{11}$$

*where* $\mathbf{a}' = (a_1, \ldots, a_{p-k})$, $a'' = \prod_{i=p-k+1}^{p} a_i$ *and* $\mathbf{d} = (d_1, \ldots, d_{p-1})$.

Several special cases are of particular interest.

*Lower Bound for $\mathcal{A}_e$–codimension*

As a consequence, Proposition 5.2 yields the minimum $\mathcal{A}_e$– codimension for germs in the contact class.

COROLLARY 5.3. *Let $V = D(F)$ be the discriminant of the versal unfolding $F : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ of an ICIS $f_0$. Suppose $k < h(D(F))$ and the weighted subspace defined by $y_1 = \ldots = y_{p-k} = 0$ is algebraically transverse to $V$ off 0. Then, the minimum $\mathcal{A}_e$–codimension of germs $g : \mathbb{C}^{n'}, 0 \to \mathbb{C}^k, 0$ in the same contact class as $f_0$ is given by*

$$\textit{minimum } \mathcal{A}_e\textit{–codimension} = \mu_k(D(F))$$

EXAMPLE 5.4. Low codimension calculations suggest that for the discriminant $D(A_n)$ of the versal unfolding of an $A_n$ singularity, the sections defined by the $n-k$ lowest weight unfolding parameters are algebraically transverse to $D(A_n)$ off 0. If this generally holds then $\mu_k(D(A_n))$ is given by (11), which can be computed to equal $\binom{n-1}{k}$. Then, by Corollary 5.3 $\binom{n-1}{k}$ is the minimum $\mathcal{A}_e$–codimension of a germ $g : \mathbb{C}^{n'}, 0 \to \mathbb{C}^k, 0$ belonging to the $A_n$ contact class.

*Free and Almost Free Hyperplane Arrangements*

Freeness plays an extremely important role for hyperplane arrangements, see [OT1]. We consider $A \subset \mathbb{C}^p$ a free hyperplane arrangement. The defining equation for $A$ is homogeneous of the form $Q = \prod \ell_i$ where $\ell_i$ are linear forms defining the hyperplanes belonging to $A$. There are homogeneous generators $\zeta_i$ for Derlog $(A)$ with $\zeta_0$ the Euler vector field, and $\zeta_i \in$ Derlog $(Q), i > 0$. Let wt $(\zeta_i) = d_i$ and let $e_i = d_i + 1$. Then, $exp(A) = (e_1, \ldots, e_p)$ are called the exponents of $A$ (note $e_1 = 1$). An almost free arrangement $\mathcal{A} \subset$ is obtained as the linear section of a free $A$. Because we are in the homogeneous case, Proposition 5.2 is always applicable. By properties of $\tau(\mathbf{D})$, we may adjust weights for the generators by decreasing all $a_i$ to zero and replacing $d_i$ by $e_i$ and compute higher multiplicities for almost free arrangements.

COROLLARY 5.5. ([**?**]Prop. 5.2]D4). *Let $\mathcal{A} \subset \mathbb{C}^p$ be an almost free arrangement based on the free arrangement $A$. If $exp'(A) = (e_2, \ldots, e_p)$, then*

$$\mu_k(\mathcal{A}) = \sigma_k(exp'(A))$$

The higher multiplicities have a close connection with other invariants of hyperplane arrangements such as the Mobius function of the arrangement evaluated at $A$, the Crapo invariant, etc [**?**]§5]D4 and [**?**]§2]D10. We single out a key property:

*Betti Numbers and Higher Multiplicities for Arrangements.*

If $\mathcal{A} \subset \mathbb{C}^p$ is a central arrangement, then the $k$-th Betti number of the complement $M(\mathcal{A}) = \mathbb{C}^p \backslash \mathcal{A}$ is given by

$$b_k(M(\mathcal{A})) \quad = \quad \mu_k(\mathcal{A}) + \mu_{k-1}(\mathcal{A}) \tag{12}$$

By this property together with Corollaries 5.5 and 3.5, we can compute algebraically the Poincaré polynomial $P(A', t)$ of the complement $M(A')$ for any almost free arrangement $A'$. Using this one can find central arrangements in $\mathbb{C}^3$, which are (automatically) locally free in the complement of 0, but which are not almost free. Hence, almost free divisors and complete intersections, like ICIS, can not be characterized by purely local conditions in the complement of a point.

As the preceding includes the free arrangement itself, we deduce as a consequence [**?**]§5]D4 the factorization theorem of Terao, which generalizes earlier results of Arnold [A1] and Brieskorn [B1] for Coxeter arrangements.

THEOREM 5.6. (Terao's Factorization Theorem [To3]). *If $A \subset \mathbb{C}^p$ is a free arrangement with exp$(A) = (e_1, \dots, e_p)$, then*

$$P(A, t) \quad = \quad \prod_{i=1}^{p} (1 + e_i t)$$

Considering critical points of functions such as $f_1^{\lambda_1} \cdots f_r^{\lambda_r}$ with the $f_i$ polynomial leads to considerably more complicated "nonlinear arrangements"which can arise by replacing hyperplanes by hypersurfaces or by intersecting a hyperplane arrangement with a smooth affine variety.

Suppose $X, 0 \subset \mathbb{C}^p$ is a homogeneous $r$–dimensional ICIS defined by a polynomial germ $f$ of multidegree $\mathbf{d} = (d_1, \dots, d_{n-r})$. Let $X_y = f^{-1}(y)$ be a global smooth Milnor fiber. If the free arrangement $A$ is transverse to $X$ off 0, then $A \cap X$ is an almost free complete intersection at 0. Moreover, for sufficiently general $y$, $A$ is transverse to $X_y$, and if both are "nondegenerate at infinity", then $A \cap X_y$ is diffeomorphic to the singular Minor fiber of $(A \cap X, 0)$ [D7] or [D10]. Then, $A \cap X_y$ defines a nonlinear arrangement on the smooth global complete intersection $X_y$. The singular vanishing cycles correspond in the real picture to "relative bounding cycles"for regions on $X_y$ determined by the nonlinear arrangement $A \cap X_y$. The generalized Lê–Greuel formula (Thm. 4.3) comnbined with Corollary 5.5 yields the number of such cycles [**?**]Theorem 8.19]D4 and [D10].

PROPOSITION 5.7. *Suppose $X \subset C^p$ is a homogeneous ICIS of multidegree $\mathbf{d}$, and the free arrangement $A$ is transverse to $X$ off 0. Then*

$$\text{number of relative bounding cycles of } (X_y, X_y \cap A) = d \cdot \sum_{j=0}^{r} s_j(\mathbf{d}-\mathbf{1})\mu_{r-j}(\mathcal{A})$$

*where $d = \prod_{i=1}^{n-r} d_i$ and $\mathbf{d}-\mathbf{1} = (d_1 - 1, \ldots, d_{n-r} - 1)$.*

The formula in Proposition 5.7 involves invariants also defined for non-free $A$. For the case of a single homogeneous hypersuface, Orlik–Terao [OT3] obtained an equivalent version of this formula but expressed in terms of the characteristic polynomial of $A$ and valid for any $A$. Having obtained the formula, it is then possible by different methods to prove that Proposition 5.7 is valid for any $A$ withour regard to freeness and extend it to hypersurface arrangements [D10].

*Remark.* Besides the number of bounding cycles, such formulas also then apply to determine the number of critical points for holomorphic functions of the form $f_1^{\lambda_1} \cdots f_r^{\lambda_r}$ which appear in hypergeometric functions [Ao], [V], [OT2], and [D7].

## 6. Relation with Buchsbaum-Rim Multiplicities

We observed that formulas such as in Proposition 5.7, although arrived at initially as special cases, may remain valid without freeness. As we relax the condition of freeness for the divisor or complete intersection $V$, we ask what form the formulas will take for the singular Milnor number or its relative version, higher multiplicities, etc. Then, lengths of modules must be replaced by more general invariants. Gaffney's work shows the importance of Buchsbaum–Rim multiplicities as invariants of modules. We note several important connections already established between singular Milnor numbers, the algebraic Buchsbaum–Rim multiplicities, and the geometric higher multiplicities.

1. First, all of the computations of singular Milnor numbers are in terms of lengths of determinantal modules on complete intersections. These modules are extended normal spaces to various groups $\mathcal{G}$ of equivalences. By results of Buchsbaum–Rim [BRm], these lengths are Buchsbaum–Rim multiplicities, which we denote by $m_{BR}$. Hence,
$$\mu_V(f_0) = m_{BR}(N\mathcal{G}_e \cdot f_0) \tag{13}$$
where for complete intersections we replace $\mu_V(f_0)$ by a relative singular Milnor number.

2. Second, Theorem 4.6 gives a general expression for $m_{BR}(M)$ in the (semi-) weighted homogeneous case, where $M$ is a determinantal module on a (possibly nonisolated) complete intersection $X, 0$.

3. Third, in §5 we associated to certain modules $M = \mathrm{Derlog}\,(V)$, geometric higher multiplicities $\mu_k(V)$. For $V, 0$ a weighted homogeneous free divisor, these are again computed as Buchsbaum–Rim multiplicities of certain related modules.

4. Fourth, in certain cases we are able to express Buchsbaum–Rim multiplicities back in terms of higher multiplicities.

When $M = \mathrm{Derlog}\,(V)$ is not free, the preceding no longer hold for lengths of the $\mathcal{K}_{V,e}$ or $\mathcal{K}_{H,e}$ normal spaces. We ask which invariants of the normal spaces will relate to the singular Milnor numbers and higher multiplicities. In particular, can we compute the (relative) singular Milnor number as the Buchsbaum–Rim multiplicity of the original $\mathcal{K}_{V,e}$ or $\mathcal{K}_{H,e}$ normal spaces for general complete intersections?

Alternately, we can consider free submodules $M' \subset \mathrm{Derlog}\,(V)$ which are "good approximations" to $\mathrm{Derlog}\,(V)$ (see §10 and §11). Can we extend the method of computing higher multiplicities for $M = \mathrm{Derlog}\,(V)$ to more general free submodules of $M$ with the same cosupport? When will these then have properties analogous to those for the geometric situation? Then, we seek relations between the invariants for the approximations, the algebraic invariants of $\mathrm{Derlog}\,(V)$, and topological invariants of $V$.

# III Topology of Singular Milnor Fibers via Logarithmic Forms

## 7. De Rham Cohomology of Free and Almost Free Divisors

Suppose $V_0, 0 \subset \mathbb{C}^n, 0$ is a divisor with reduced defining equation h. There are several natural questions concerning the topology of $(V_0, 0)$ and its complement which can possibly be addressed using differential forms. These include computing the cohomology of:

1. the complement $\mathbb{C}^n \backslash V_0$;

2. the smooth Milnor fiber of $h$;

3. the singular Milnor fiber of an almost free divisor $V_0$ based on the free divisor $V, 0$.

In the case of 2) and 3), this would provide a basis as in [B2] and [Gr] for introducing a Gauss–Manin connection.

For 1), by general results of Grothendieck [Grk] and Griffiths [Grf] the cohomology of the complement of a divisor can be computed using the complex of meromorphic forms on the complement. For Coxeter arrangements, Arnold [A1] and Brieskorn [B1] computed the cohomology of the complement of the arrangement using logarithmic forms. This was extended to general hyperplane arrangements, by Orlik–Solomon [OS1] building on Brieskorn's ideas. For an arrangement $A = \cup H_i \subset \mathbb{C}^n$, let $H_i$ be defined by the linear form $\ell_i$. These results express the cohomology of the complement as the quotient of an exterior algebra $R(A)$ on generators $\omega_i = d\ell_i/\ell_i$ by the ideal generated by the relations $r_{ijk} = \omega_i \wedge \omega_j + \omega_j \wedge \omega_k + \omega_k \wedge \omega_i$, for triples of $H_i$ with $\operatorname{codim}(H_i \cap H_j \cap H_k) = 2$.

Castro, Mond and Narvaez [CMN] show that this result has a natural analogue for free divisors using the complex of logarithmic forms. They consider free divisors which are locally weighted homogeneous (i.e. locally weighted homogeneous at each point for some choice of local coordinates).

THEOREM 7.1. ([CMN]). *Let $V, 0 \subset \mathbb{C}^p$ be a free divisor which is locally weighted homogeneous, then for all $k$*

$$H^k(\mathbb{C}^p \backslash V : \mathbb{C}) \simeq H^k(\Omega^\bullet(logV))$$

Unfortunately, in general this result does not even extend to the simplest almost free divisors such as isolated hypersurface singularities $V, 0 \subset \mathbb{C}^n, 0$. Holland and Mond [HMo] define an obstruction, the "logarithmic defect", $\delta(logV)_0$, which equals $\dim Gr_n^W H^{n-1}(F; \mathbb{C})$ for the weight filtration for the mixed Hodge structure on the Milnor fiber $F$ of $V$ [St]. If $\delta(logV)_0 \neq 0$, the analogue of Theorem 7.1 fails. They show using results of Steenbrink [St] that even for the simple singularities $A_k$, $D_k$, and $E_k$, there are values of $n$ and $k$ for which it fails.

For a free divisor $V, 0 \subset \mathbb{C}^p$ with singular set $Z = \operatorname{sing}(V)$, Alexandrov has obtained several results computing local cohomology using $\Omega^\bullet(\log V)$ as described in [Av2]. He computes the Poincaré polynomials of $H^\bullet_Z(\Omega^q_V)$ and $H^\bullet_Z(\Omega^q(\log V))$ for weighted homogeneous $V$. Second, he applies ideas of Kunz to obtain a form of Grothendieck local duality for the local cohomology of free divisors [?]Thm. 2.3]Av2, obtaining a perfect pairing between the infinite dimenional spaces $T^1_V$ and the torsion module Tors $\Omega^1_V$ , both of which are isomorphic to $\mathcal{O}_{\mathbb{C}^p,0}/J(h)$ (with $J(h)$ denoting the Jacobian ideal of $h$). Third, he relates $\Omega^\bullet(\log V)$ to the complex of regular meromorphic differential forms $\omega_V^{\bullet -1}$, obtaining an exact sequence [?]§4]Av2 generalizing the classical sequence involving the Poincaré residue for a smooth divisor.

$$0 \longrightarrow \Omega^q_{\mathbb{C}^p,0} \longrightarrow \Omega^q(\log V) \xrightarrow{\text{res}} \omega_V^{q-1} \longrightarrow 0 \qquad (14)$$

As a result of (14) he also obtains the Poincaré polynomial of $\omega_V^{q-1}$. In the special case of an isolated hypersurface germ $V, 0$ of dimension $\geq 2$, he relates $\mu$ and $\tau$ to the dimensions of the cohomology of the complex of sections $H^*(\Omega^\bullet(\log V))$ to obtain [**?**]Cor. 3]Av2

$$\dim_{\mathbb{C}} H^p(\Omega^\bullet(\log V) - \dim_{\mathbb{C}} H^{p-1}(\Omega^\bullet(\log V) = \mu - \tau$$

exhibiting the close relation between $H^*(\Omega^\bullet(\log V))$ and other invariants of $V$.

*de Rham Cohomology of the Singular Milnor Fiber*

¿From our point of view, an especially important result was obtained by Mond [Mo3]. He identifies the correct way to use $\Omega^\bullet(\log V)$ to modify the complex of Kähler forms on an almost free divisor to obtain a complex of forms which computes the cohomology of the singular Milnor fiber.

For a divisor $V, 0 \subset \mathbb{C}^p, 0$ defined by $h$, we have the complex of Kähler forms

$$\Omega^\bullet_V = \Omega^\bullet_{\mathbb{C}^p,0} / (h\Omega^\bullet_{\mathbb{C}^p,0} + dh \wedge \Omega^{\bullet-1}_{\mathbb{C}^p,0}) \qquad (15)$$

Mond observed that $\Omega^\bullet_V$ has torsion and that the torsion can be identified as $h\Omega^\bullet(\log V)/(h\Omega^\bullet_{\mathbb{C}^p,0} + dh \wedge \Omega^{\bullet-1}_{\mathbb{C}^p,0})$. This led him to define $\check{\Omega}^\bullet_V = \Omega^\bullet_V / h\Omega^\bullet(\log V)$. Then, Mond observes that by Theorem 2.3, the complex $\check{\Omega}^\bullet_V$ has properties analogous to those of $\Omega^\bullet(\log V)$. Specifically, exterior differentiation induces a differential on $\check{\Omega}^\bullet_V$ so it is a complex; and it is preserved under both Lie deriviative by and inner product with vector fields in $\mathrm{Derlog}\,(V)$. If $V, 0$ is a free divisor, although $\check{\Omega}^k_V$ is no longer free, he shows $\check{\Omega}^k_V$ is a maximal Cohen–Macaulay module. Moreover, Mond shows

PROPOSITION 7.2. *For a divisor $V, 0 \subset \mathbb{C}^p, 0$, $\check{\Omega}^k_V = 0$ for $k \geq p$; and if $V, 0$ is locally weighted homogeneous, then $\check{\Omega}^\bullet_V$ is a resolution of $\mathbb{C}_V$. Hence, if $U$ is a Stein open subset and $V \subset U$ is locally weighted homogeneous at each point, then $H^k(\Gamma(\check{\Omega}^\bullet_V)) = H^k(V; \mathbb{C})$ for all $k$.*

To transfer this structure to almost free divisors, we note that a defining germ $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ for an ICIS is naturally a smoothing, so for $y \notin D(f)$, $f_0^{-1}(y) \cap B_\epsilon$ is the smooth Milnor fiber. In our case of a nonlinear section $f_0$ of $V$ defining the AFD $(V_0, 0)$, we must instead consider a stabilization of $f_0$. Let $f : \mathbb{C}^{n+q}, 0 \to \mathbb{C}^p, 0$ be a deformation which is a submersion at 0 (Mond considers other "admissible"deformations, and then further extends them to obtain such a deformation; however, the singular Milnor fibers will be the same). Then, by e.g. the versality theorem for $\mathcal{K}_V$ equivalence, $\mathcal{V} = f^{-1}(V), 0 \subset \mathbb{C}^{n+q}, 0$ is diffeomorphic to $V \times \mathbb{C}^r, 0$, for $r = n - p + q$. If $n < hn(V)$, then algebraic and geometric transversality are the same. Hence, by the parametrized transversality theorem, for almost all values $u \in \mathbb{C}^q$, $f_u(x) = f(x, u)$ is algebraically transverse to $V$. The set of values $u$ where $f_u$ fails to be transverse form the $\mathcal{K}_V$–discriminant $D_V(f)$, see §8 (Mond calls this the logarithmic discriminant).

Let $\pi : \mathcal{V}, 0 \to \mathbb{C}^q, 0$, be the restriction to $\mathcal{V}$ of the projection $\mathbb{C}^{n+q}, 0 \to \mathbb{C}^q, 0$; this is the analogue of the ICIS germ $f_0$. For $\epsilon > 0$ and $u \notin D_V(f)$ both sufficiently small, $\pi^{-1}(u) \cap B_\epsilon = f_u^{-1}(V) \cap B_\epsilon$ is a singular Milnor fiber for $f_0$. Then, Mond introduces the complexes which serve as analogues of those for ICIS (compare [Gr] or [Lo])

$$\begin{aligned} \check{\Omega}^\bullet_{V_0} &= \Omega^\bullet_{\mathbb{C}^n,0} / \langle f_0^*(h\Omega^\bullet(\log V)) \rangle \\ \check{\Omega}^\bullet_{\mathcal{V}/\mathbb{C}^q} &= \Omega^\bullet_{\mathbb{C}^{n+q},0} / \left( \langle f^*(h\Omega^\bullet(\log V)) \rangle + \sum du_i \wedge \Omega^{\bullet-1}_{\mathbb{C}^{n+q},0} \right) \end{aligned} \tag{16}$$

where "$\langle M \rangle$"now denotes the ideal in the exterior algebra generated by $M$. Moreover, by applying the analogue of the Poincaré Lemma [**?**]Lemma 3.10]Mo3 when $f$ is transverse to $V$, we can replace $\langle f^*(h\Omega^\bullet(\log V)) \rangle$ by $\langle (h \circ f\Omega^\bullet(\log \mathcal{V})) \rangle$.

There are three crucial properties which Mond establishes:

1. $$R^k\pi_*(\mathbb{C}_\mathcal{V}) \otimes_\mathbb{C} \mathcal{O}_{\mathbb{C}^q} \simeq \mathcal{H}^k(\pi_*(\check{\Omega}^\bullet_{\mathcal{V}/\mathbb{C}^q})) \qquad \text{outside } D_V(f)$$

2. there is an isomorphism of stalks at 0,
$$\mathcal{H}^k(\pi_*(\check{\Omega}^\bullet_{\mathcal{V}/\mathbb{C}^q}))_0 \simeq \pi_*(\mathcal{H}^k(\check{\Omega}^\bullet_{\mathcal{V}/\mathbb{C}^q}))_0 \qquad \text{for all } k > 0;$$

3. the sheaves $\mathcal{H}^k(\pi_*(\check{\Omega}^\bullet_{\mathcal{V}/\mathbb{C}^q}))$ are coherent for all $k$.

The coherence property 3) is crucial for providing a basis of local sections which generate $\mathcal{H}^k(\pi_*(\check{\Omega}^\bullet_{\mathcal{V}/\mathbb{C}^q}))_u$ at $u \in D_V(f)$. By 1) this explicitly gives a basis for the de Rham cohomology of the singular Milnor fiber over $u$. The earlier results of Brieskorn and Greuel used the extension of the germ to a proper mapping to obtain coherence. Mond instead applies a result of Van Straten [VS2]. Then, by an argument which generally follows that for ICIS given in [**?**]Cor. 8.8]Lo, Mond proves

THEOREM 7.3. *Let $V_0, 0 \subset \mathbb{C}^n, 0$ be an almost free divisor defined by a nonlinear section $f_0$ of $V, 0$. Then, the deRham cohomology of a singular Milnor fiber at $u$ for a stabilization $f$ is computed as the fiber of the sheaf $\mathcal{H}^*(\pi_*(\check{\Omega}^{\bullet}_{\mathcal{V}/\mathbb{C}^q}))$ at $u$. It is nonzero only in dimension $n-1$ and its rank is given by the rank of the stalk at 0, $\dim_{\mathbb{C}} H^{n-1}(\pi_*(\check{\Omega}^{\bullet}_{\mathcal{V}/\mathbb{C}^q})_0)$.*

The earlier properties imply that $\check{\Omega}^{p-1}_{V_0}$ has torsion and, Theorem 7.3 allows the singular Milnor number to be computed in the weighted homogeneous case as $\dim Tor\check{\Omega}^{p-1}_{V_0}$.

The singular Milnor fiber can be triangulated, so the vanishing cycles can be represented by a union of simplices. Hence, we can still integrate forms over these cycles. This then leads to Mond's construction of the Gauss–Manin connection for a stabilization of an almost free divisor with properties analogous to those of ICIS, especially that it has a regular singularity along the $\mathcal{K}_V$–discriminant [**?**]§6]Mo3. This allows Mond to extend a number of formulae valid for ICIS [Gr] to the singular Milnor fiber. These results allow for the possibility that many of the earlier results can be extended explicitly using De Rham cohomology.

# IV Discriminants

## 8. Discriminants for Deformations of Sections

Among the original examples of free divisors in Theorem 2.2, are discriminants for versal unfoldings of ICIS, and bifurcation sets for isolated hypersurface singularities. We explore the general mechanism which produces free divisors in many different situations, which includes both of these examples. As mentioned earlier, there are criteria of Saito and Alexandrov for verifying the a hypersurface singularity is a free divisor. In addition, there is a criterion of Goryunov [Go2] for functions on space curves: $\mu = \tau$ implies the discriminant of the versal unfolding is a free divisor. Mond and Van Straten then showed that $\mu = \tau$ always holds for functions on space curves, implying that the discriminant is always a free divisor. Van Straten [VS2] had earlier shown, using Saito's criterion, that the discriminants of versal defomations of space curves are free divisors.

Using a different approach and Van Straten's result, it was shown in [D6] that the bifurcation set for smoothings of space curves are free divisors. In fact, this last result applies a general method based on representing various maps, unfoldings, etc. as sections of varieties $V$ and giving sufficient conditions that the $\mathcal{K}_V$–discriminant for the versal unfolding of such a section is a free divisor. The approach is summarized.

**Freeness Principle for Discriminants**

$$\begin{array}{rcl} \text{Cohen–Macaulay of codim 1} & + & \text{Genericity of Morse Type} \\ \text{Singularities} & \Longrightarrow & \text{Freeness of Discriminants} \end{array} \tag{17}$$

We explain how this principle applies to a wide variety of situations and discuss what happens as each condition fails.

Given $V, 0 \subset \mathbb{C}^p, 0$ and a germ $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ which has finite $\mathcal{K}_V$–codimension, let $F : \mathbb{C}^{n+q}, 0 \to \mathbb{C}^{p+q}, 0$ be a $\mathcal{K}_V$–versal unfolding of $f_0$. Using local coordinates $u$ for $\mathbb{C}^q$, we write $F(x, u) = (\bar{F}(x, u), u)$, and denote $f_u(x) = \bar{F}(x, u) : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ as a function of $x$ . Also, we denote by $\pi$ the projection $\mathbb{C}^{n+q}, 0 \to \mathbb{C}^q, 0$.

DEFINITION 8.1. *The $\mathcal{K}_V$–critical set of $F$, $C_V(F)$, consists of points $(x_0, u_0)$, such that the germ $f_{u_0}$ is not algebraically transverse to $V$ at $x_0$. It can equivalently be defined as*

$$C_V(F) = supp(N\mathcal{K}_{V,un,e} \cdot F)$$

*(where $N\mathcal{K}_{V,un,e} \cdot F$ is the extended normal space for the action of the unfolding group $\mathcal{K}_{V,un}$). The $\mathcal{K}_V$–discriminant of $F$ is then defined to be $D_V(F) = \pi(C_V(F))$.*

We apply Teissier's method [Te2] for associating a non–reduced structure to the discriminant via the 0–th Fitting ideal.

It can be shown [**?**]§2]D7 that as $f_0$ has finite $\mathcal{K}_V$–codimension, $\pi|C_V(F)$ is finite to one. Hence, by Grauert's theorem $D_V(F)) = \pi(C_V(F))$ is the image of an analytic subset $C_V(F)$ under a finite map, hence is also an analytic germ of the same dimension as $C_V(F)$. Second, by [**?**]Prop. 2.4 and Cor. 2.5]D7 (Mond gives an equivalent formulation), we obtain

PROPOSITION 8.2. *If $V$ is a free divisor and $F$ is a $\mathcal{K}_V$–versal unfolding of $f_0$ (or at least $\bar{F}$ is algebraically transverse to $V$ at $0$) and $n < hn(V)$, then, both $C_V(F)$ and $D_V(F)$ are Cohen–Macaulay of dimension $q - 1$.*

This is half of the condition (17). To obtain the other half ,we define

DEFINITION 8.3. *We say that a free divisor $V, 0 \subset \mathbb{C}^p, 0$, generically has Morse–type singularities in dimension $n$ if: all points on canonical stata of $V$ of codimension $\leq n + 1$ have Morse singularities of nonzero exceptional weight type (which we do not define here); and any stratum of codimension $> n + 1$ lies in the closure of a stratum of codimension $= n + 1$.*

This condition combined with Proposition 8.2 allows us to apply Saito's criterion to obtain a general criterion for freeness of $\mathcal{K}_V$–discriminants.

THEOREM 8.4. ([**?**]Thm. 2]D7). *Let $V, 0 \subset \mathbb{C}^p, 0$ be a free divisor which generically has Morse–type singularities in dimension $n$ where $n < hn(V)$. Then, the $\mathcal{K}_V$–discriminant of the versal unfolding for any $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ is a free divisor. Moreover,*

$$\mathrm{Derlog}\,(D_V(F)) = \textit{module of } \mathcal{K}_V\textit{–liftable vector fields.}$$

By a $\mathcal{K}_V$–liftable vector field $\eta \in \theta_q$ we mean there are

$$\xi \in \mathcal{O}_{\mathbb{C}^{n+q},0}\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\} \quad \text{and} \quad \zeta \in \mathcal{O}_{\mathbb{C}^{n+q},0}\{\zeta_1, \ldots, \zeta_p\}$$

satisfying:

$$(\xi + \eta)\,(\bar{F}) \quad = \quad \zeta \circ \bar{F} \tag{18}$$

## 9. Morse-type Singularities for Sections and Mappings on Divisors

To apply Theorem 8.4, we must verify for a given $V$ the genericity of Morse–type singularities. We first illustrate the applicability in the case of bifurcation sets of finitely $\mathcal{A}$–determined map germs $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$. By (1.1), $f_0$ is obtained as the pullback of the stable unfolding $F$ of $f_0$ by an embedding $g_0$. Furthermore, by Theorem 1.5, $f_0$ is a stable multigerm over a point $y \in D(F)$ iff $g_0$ is transverse to $D(F)$ at $y$. Hence, for an unfolding $f(x, v) = (\bar{f}(x, v), v)$ of $f_0$ induced by an unfolding $g(y, v) = (\bar{g}(y, v), v)$ of $g_0$, $f_v(x) = \bar{f}(x, v)$ is stable iff $g_v(x) = \bar{g}(x, v)$ is transverse to $D(F)$. Thus, the bifurcation set of $f$ is the $\mathcal{K}_{D(F)}$–discriminant of $g$. Thus, it is sufficient to determine when the discriminants $D(F)$ generically have Morse–type singularities. Using the results on $\mathcal{A}_e$–codimension 1 germs and multigerms, we can identify the following class [**?**]§6]D7.

DEFINITION 9.1. *We say that a finitely $\mathcal{A}$–determined (multi)germ $f_0 : \mathbb{C}^n, S \to \mathbb{C}^p, 0$ with $n \geq p$ belongs to the distinguished bifurcation class of (multi)germs if it satisfies one of the following :*

1. *"general case": $n \neq p + 1$ and $p \leq 4$;*
2. *"worse case": $n = p + 1$ and $p \leq 3$;*

*3. "best cases":*

*(i) corank 1 (multi)germs such that :*

*(a) $n = p + 1$ and $p \leq 6$ or*

*(b) $n > p + 1$ and $p \leq 5$*

*(ii) $\Sigma_{n-p+1}$ and $\Sigma_{2,(1)}$ (multi)germs without restriction on $n \geq p$.*

For example, the $\Sigma_{n-p+1}$ germs are germs in the $\mathcal{K}$–equivalence class of the $A_k$ germs for $n \geq p$; and the $\Sigma_{2,(1)}$ germs are those in the $\mathcal{K}$–equivalence class of $I_{2,b}$, consisting of the germs $f_0(x, y) = (xy, x^2 + y^b)$ for $n = p$.

Then, the Terao and Bruce result extends as follows [**?**]Thm. 3]D7.

THEOREM 9.2. *Let $f_0 : \mathbb{C}^n, S \to \mathbb{C}^p, 0$ be a finitely $\mathcal{A}$–determined (multi) germ which belongs to the distinguished bifurcation class. Then, the bifurcation set of its $\mathcal{A}$–versal unfolding is a free divisor.*

David Mond and Andrew DuPlessis obtained an equivalent version of this by directly using $\mathcal{A}$–equivalence.

Also, $\{0\} \subset \mathbb{C}^p$ generically has Morse–type singularities (i.e. the usual Morse singularities) which yields for $p = 1$ the freeness of (the usual) discriminants of versal unfoldings of isolated hypersurface singularities. There are a number of other examples of free divisors which generically have Morse–type singularities. These are given in [**?**]§6-9]D7, leading to the freeness of various discriminants and bifurcation sets.

One revealing example is the Manin–Schechtman discriminantal arrangement which naturally extends the braid arrangement [MS]. For a central general position hyperplane arrangement $A = \cup_{i=1}^{n} H_i \subset \mathbb{C}^k$, it is obtained as the pullback of the Boolean arrangement $A_n \subset \mathbb{C}^n$ by a linear embedding $\varphi$. The associated discriminantal arrangement $B(n, k)$ consists of the set of normal translation vectors for the hyperplanes $H_i$ which when applied, do not give a general position arrangement. However, unlike the braid arrangement, $B(k + 3, k)$ is not free when $k \geq 2$ Orlik–Terao [**?**]Prop. 5.6.6]OT1.

COROLLARY 9.3. *Let $A \subset \mathbb{C}^k$ be a central general position arrangement defined by $\varphi : \mathbb{C}^k \to \mathbb{C}^n$. Then, the $\mathcal{K}_{A_n}$–discriminant of the versal unfolding of $\varphi$ is a free divisor.*

Then, $B(n, k)$ is the intersection of $D_{A_n}(\varphi)$ with the linear subspace of translationdeformations and the remaining unfolding parameters (at least for low $(n, k)$) give equisingular $\mathcal{K}_{A_n}$–deformations. Thus, $D_{A_n}(\varphi)$ is topologically equivalent to a suspension of $B(n, k)$, even though $D_{A_n}(\varphi)$ is free while $B(n, k)$ is not.

EXAMPLE 9.4. In the opposite direction, an example of a free divisor without a Morse–type singularity is given by the free hyperplane arrangement $A \subset \mathbb{C}^3$ defined by $Q = xyz(x-y)$. A Morse–type singularity of dimension 2, if it existed, would be obtained as the inclusion of a generic hyperplane section. However, this gives an arrangement of 4 lines in $\mathbb{C}^2$ with singular Milnor fiber having 2 singular cycles (see [**?**]§4, 9]D7). This corresponds to the section having $\mathcal{K}_{A,e}$–codimension 2.

These methods can be adjusted to apply to the dual case of the equivalence of mappings $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ fixing a divisor $V, 0 \subset \mathbb{C}^n, 0$ in the source. We denote the contact group preserving $V$ by ${}_V\mathcal{K}$. For certain special free divisors $V$ and simple functions $f_0$, Arnold [A3], Lyaschko [Ly], Zakalyukin [Z], and Goryunov [Go2] prove that the discriminant is a free divisor. What is surprising is that this fails in general even with $V$ free, and it does not depend on whether we consider functions or ICIS map germs. Moreover, the freeness of the ${}_V\mathcal{K}$–discriminant depends on whether generically $V$ has Morse type singularities. This is because the "Morse type singularities"for ${}_V\mathcal{K}$–equivalence can be canonically identified with the Morse type singularities for $\mathcal{K}_V$–equivalence [**?**]§6]D9. An equivalent version of example (9.3) shows that the discriminant will not be free for the module of ${}_V\mathcal{K}$–liftable vector fields. We do obtain [**?**]§6]D9 (with consequences for ${}_V\mathcal{A}$ and ${}_V\mathcal{R}$)

THEOREM 9.5. *If $V, 0 \subset \mathbb{C}^n, 0$ is a free divisor which generically has Morse–type singularities in dimension $n-p$, then the ${}_V\mathcal{K}$–discriminant of the ${}_V\mathcal{K}$–versal unfolding of an ICIS germ $f_0$ (of finite ${}_V\mathcal{K}$–codimension) is a free divisor with*

$$\operatorname{Derlog}(D_{{}_V\mathcal{K}}(F)) = \textit{module of } {}_V\mathcal{K}\textit{–liftable vector fields.}$$

## 10. Beyond Freeness - Free* Divisor Structures

When either of the two conditions in (17) fails for some group of equivalences, we can no longer conclude that the appropriate discriminants are free. We explain how we must introduce new notions reflecting the weaker structure that does remain. We restrict our remarks to the specific question of whether the module of liftable vector fields defines a free divisor structure, reflecting the intrinsic features of the equivalence. There are always cases such as isolated curve singularities in $\mathbb{C}^2$ which are always free by [Sa], unrelated to how they are obtained.

First we relax the condition of genericity of Morse type singularities. For example (9.4), the $\mathcal{K}_A$–discriminant has a nonreduced structure resulting from the generic singularities having $\mathcal{K}_{A,e}$–codimension 2 [**?**]§9]D7. If the discriminant is Cohen–Macaulay of codimension 1 then at least the discriminant has a "free* divisor"structure defined by the liftable vector fields.

DEFINITION 10.1. *By a hypersurface germ $V, 0 \subset \mathbb{C}^p$ having a free* divisor structure on $\mathbb{C}^{p'}$, where $p' = p + m \geq p$, we shall mean: for $V' = V \times \mathbb{C}^m \subset \mathbb{C}^{p'}$ we are given an $\mathcal{O}_{\mathbb{C}^{p'},0}$– submodule* $\mathrm{Derlog}^*(V) \subseteq \mathrm{Derlog}(V')$ *which satisfies:*

1. $\mathrm{Derlog}^*(V)$ *is a free $\mathcal{O}_{\mathbb{C}^{p'},0}$–module of rank $p'$; and*

2. *supp*$(\theta_{p'}/\mathrm{Derlog}^*(V)) = V'$

In fact there are many ways, including trivial ones, of putting a free* divisor structure on certain hypersurfaces. Their usefulness depends on certain measures of nontriviality and on the intrinsic properties of $\mathrm{Derlog}^*(V)$. By suspending $V$ we can assume $\mathrm{Derlog}^*(V) \subset \mathrm{Derlog}(V)$. This raises the question of how one can understand properties of nonfree submodules of $\theta_p$ or more generally $\mathcal{O}^p_{\mathbb{C}^n,0}$ using free submodules which are in an appropriate sense a "good approximation".

For example, we know isolated surface singularities $V, 0 \subset \mathbb{C}^3, 0$ are never free divisors, but are always almost free divisors. Although nonisolated surface singularities need not be free, any weighted homogeneous surface singularity has a natural "pfaffian " free* divisor structure [**?**]§1]D8.

Also, by the proof of Theorem 8.4 with $n < hn(V)$, but without having genericity of Morse–type singularities, the $\mathcal{K}_V$ discriminant is a free* divisor defined by $\mathrm{Derlog}(V)$ [**?**]Thm. 1]D8. For the various consequences mentioned in Example (9), if the conditions are relaxed so genericity of Morse type singularities fails, then at least the discriminants are free* divisors defined by the modules of liftable vector fields. For example, all finitely $\mathcal{A}$–determined germs in the nice dimensions (in the sense of Mather [M-IV]) have bifurcation sets which are free* divisors [**?**]§1]D8 for the module of $\mathcal{A}$–liftable vector fields.

Importantly, we can still compute the vanishing topology for free* divisors using a modification of Theorem 3.3, where we must correct for "virtual singularities"for the $\mathrm{Derlog}^*(V)$ structure [**?**]§4]D8.

## 11. Cohen-Macaulay Reductions for Groups of Equivalences

If we relax instead the first condition in (17), then the group $\mathcal{G}$ of equivalences does not have the correct algebraic structure to ensure that the discriminant is Cohen–Macaulay of codimension 1.

EXAMPLE 11.1. Several examples where this occurs are for sections of free complete intersections of codimension $\geq 2$ (with the exception of $\{0\} \subset \mathbb{C}^p$), the relative case of the intersection of an ADF on an AFCI, and the case of functions on complete intersections where either we allow both the function and complete intersection to deform or we fix the complete intersection.

In these cases which are considered in [D8] and [D9], we introduce a "Cohen–Macaulay reduction"of the original group, which allows us to keep the same discriminant but with a different structure.

DEFINITION 11.2. *Given a geometric subgroup $\mathcal{G}$ of $\mathcal{A}$ or $\mathcal{K}$ which has geometrically defined discriminants, a Cohen–Macaulay reduction of $\mathcal{G}$ (abbreviated CM–reduction) consists of a geometric subgroup $\mathcal{G}^* \subset \mathcal{G}$ which still acts on $\mathcal{F}$ (and $\mathcal{F}_{un}$) such that:*

1. *$\mathcal{G}^*$ is Cohen–Macaulay (i.e. for a $\mathcal{G}^*$–versal unfolding $F$ on $q$ parameters, the normal space $N\mathcal{G}^*_{une} \cdot F$ viewed as $\mathcal{O}_{\mathbb{C}^q,0}$–module is Cohen–Macaulay, and whose support, the discriminant $D_{\mathcal{G}^*}(F)$, has codimension 1);*

2. *$f \in \mathcal{F}$ has finite $\mathcal{G}^*$–codimension iff it has finite $\mathcal{G}$–codimension;*

3. *if $F$ is a $\mathcal{G}^*$–versal unfolding on $q$ parameters, then as $\mathcal{O}_{\mathbb{C}^q,0}$–modules,*

$$supp(N\mathcal{G}^*_{un,e} \cdot F) \quad = \quad supp(N\mathcal{G}_{un,e} \cdot F) \tag{19}$$

As far as we have determined, this is not a reduction in the sense of Rees [Re], [KR] and Gaffney [GK]. In earlier results, the RHS in Theorems 4.2, 4.3 and 4.4 are normal spaces for CM–reductions.

We then establish two key results regarding CM-reduction [**?**]Thm. 2 and 3]D9.

THEOREM 11.3. *Suppose a group $\mathcal{G}$ has a CM–reduction, then*

1. *the $\mathcal{G}$–discriminants for $\mathcal{G}$–versal unfoldings are free* divisors for the module of $\mathcal{G}^*$–liftable vector fields; and*

2. *provided $\mathcal{G}$ generically has Morse–type singularities which are $\mathcal{G}^*$–liftable, the $\mathcal{G}$ –discriminants are free divisors.*

In particular, for the examples in (11.1), either they have CM–reductions or are Cohen–Macaulay. Hence, their corresponding $\mathcal{G}$–discriminants are free* divisors. Moreover, we may apply the theorem and conclude that the corresponding $\mathcal{G}$–discriminants of versal unfoldings are free divisors for the following: ICIS, which are sections of $\{0\} \subset \mathbb{C}^p$ (recovering Looijenga's result); the relative cases of the intersection of an ADF on an ICIS; functions on ICIS where we allow both the functions and ICIS to deform (related to a result of Mond–Montaldi [MM]); or a complete intersection with boundary singularity a free divisor, with appropriate restrictions on the free divisor generically having Morse–type singularities (9.5).

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