SMOOTHNESS AND GEOMETRY OF BOUNDARIES ASSOCIATED TO SKELETAL STRUCTURES I: SUFFICIENT CONDITIONS FOR SMOOTHNESS

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Introduction

There are a number of constructions which begin with a (piecewise) smooth object and associate to it a singular set which contains information regarding the relative position, geometry and shape for the original object. Examples are the "conflict set"(Maxwell set) associated to a parametrized family of smooth functions, the Voronoi set (skeleton) associated to a collection of regions with (piecewise) smooth boundaries, the caustic set for wave front evolution, and the shock set for hyperbolic equations. Several of these methods are used for analyzing shapes in computer imaging and vision, e.g. the Blum medial axis [BN], chordal locus of Brady and Asada, [BA] and arc-segment medial axis of Leyton[Le].

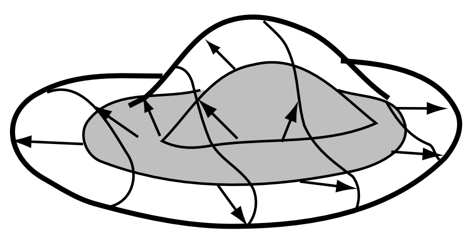

FIGURE 1. Skeletal Structure and Associated Boundary

Of these the prominent method introduced by Blum identifies the locus of centers of circles in 2D (or spheres in 3D) which are contained in the object and are tangent in at least two points (or having a single degenerate tangency). This locus is called the Blum medial axis (also called the "central set"in mathematics literature, see Yomdin [Y]). It can alternately be described as the Maxwell set (conflict set) for the family of distance functions on the boundary as in Mather [M2], or the shock set for the "grassfire flow"as in Kimia-Tannenbaum-Zucker [KTZ]. Weakening the inclusion requirement leads to a more general "symmetry set"of Bruce–Giblin–Gibson [BGG]. The definitions naturally extend to higher dimensions.

The medial axis is a prototype for the examples we consider. In the generic case it is a Whitney stratified set on which are defined associated geometric structures

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containing additional geometric information. There is a multivalued radial vector field U from points of the medial axis M to the corresponding points of tangency on the boundary and the radial distance function $r = \|U\|$. As well there are differential geometric properties of M; and in 2D and 3D there is associated a natural frame field on M defined using a normal vector field and the gradient of r (see [P3]).

To treat the medial structure (i.e. medial axis and its associated structure(s)) as an intrinsic object of interest, we must be able to perform operations on it and obtain medial–like structures which have associated smooth boundaries for which we can deduce geometric properties. For example, we would like: to replace the medial axis by an approximation (using e.g. Kimia et al [KTZ], Szekley et al [SN], or Siddiqi et al [SB]); to be able to deform the medial structure (rather than the entire object) for comparison with that of another object; to replace the medial axis by a simpler structure which emphasizes certain features (e.g. the "M-rep" representation of Pizer and coworkers [P1]); or in combination with statistical methods for medical and biological problems, to determine for a population of objects an "average medial structure" which exhibits averages and principal components of properties of the individual medial structures, and from which they can be obtained as perturbations [P2].

However, there is no guarantee that beginning with medial structures of smooth boundaries, that the resulting structure will be a medial structure of an associated boundary, nor that the associated boundary need be smooth (see Fig. 2). Furthermore, we ask how can we determine the geometric properties of the resulting associated boundary? In this paper our goal is to give answers to these questions.

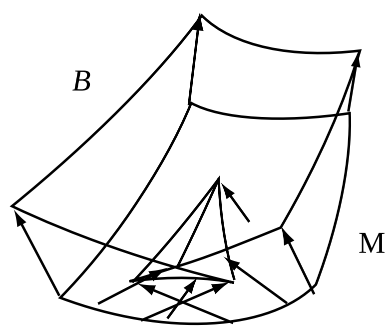

FIGURE 2. Failure of Smoothness for the Boundary Associated to a Skeletal Structure

We introduce as a fundamental object a *skeletal structure* (M, U) in $\mathbb{R}^{n+1}$. It consists of an n–dimensional skeletal set $M \subset \mathbb{R}^{n+1}$ (which will be a special type of Whitney stratified set) and multivalued vector field U on M. We relax many of the conditions usually satisfied by Blum medial axes such as, for example, not requiring that all vectors from a point in M have the same length; nor do we require that M only exhibit the properties normally exhibited by generic medial axes.

Associated to such a skeletal structure we define an associated boundary $\mathcal{B}$ via a radial map on M. To determine the smoothness and geometric properties of $\mathcal{B}$, we define for a skeletal structure a *radial shape operator* S_{rad} and on edge points of M an *edge shape operator* S_E. A compatibility condition for U is introduced using a *compatibility* 1–*form* η_U which relates the direction of U with the gradient of its magnitude r (as U is multivalued so is η_U). The shape operators we define are not shape operators in the usual differential geometric sense but rather measure how U bends relative to M, resp. ∂M, without explicitly introducing the differential geometry of M or ∂M. Our shape operators need not be self-adjoint.

In this first part of the paper, we provide (necessary and) sufficient conditions in terms of the radial and edge shape operators and a compatibility condition that the boundary $\mathcal{B}$ is smooth. In the second part [D1], we use the same shape operators to determine in the Blum case the differential geometric shape operator for the boundary. This allows us to compute all of the intrinsic geometry of the boundary in terms of the medial data. In the last part [D2], we apply the results to give for 1D and 2D medial axes, specific formulas for the geometry of the boundary of objects in terms of medial data. In particular, we define on the medial axis using only the unit radial vector field a "geometric medial map"which reveals the main geometric features of the boundary at the corresponding points. This includes both intrinsic differential geometry and geometry relative to the medial axis and allows comparison of boundary geometry purely from medial data.

The sufficient conditions for smoothness involve three conditions. For the first, we let $\kappa_{r\,i}$ denote the *principal radial curvatures* which are the eigenvalues of the radial shape operator. Second, we also consider the *principal edge curvatures* $\kappa_{E\,i}$ which are generalized eigenvalues of the edge shape operator.

We consider a skeletal structure (M, U) which satisfies the following three conditions :

(1) (*Radial Curvature Condition*) For all points of M off ∂M

$$r < \min\{\frac{1}{\kappa_{r\,i}}\} \quad \text{for all positive principal radial curvatures } \kappa_{r\,i}$$

(2) (*Edge Condition*) For all points of $\overline{\partial M}$ (closure of ∂M)

$$r < \min\{\frac{1}{\kappa_{E\,i}}\} \quad \text{for all positive principal edge curvatures } \kappa_{E\,i}$$

(3) (*Compatibility Condition*) For all singular points of M (which includes edge points), $\eta_U \equiv 0$.

The Radial Curvature, Edge, and Compatibility Conditions involve choices of values for U and hence are multi-valued conditions at each point. In the radial curvature condition it is to be understood that the r value associated to a given value of U satisfies the inequality for the shape operator associated to that value. Thus, at smooth points of M, we have inequalities corresponding to each side of M.

Then, we show (Theorem 2.5) that the associated boundary $\mathcal{B}$ is an immersed topological manifold which is smooth at all points except possibly those corresponding to singular points of M (the set of singular points is denoted M_{sing}); and at those points $\mathcal{B}$ is weakly C^1 (which means that it has a unique well-defined limiting tangent plane at these points – this implies it is C^1 at points coming from strata of M of codimension 1). Furthermore, the map from smooth points of $\mathcal{B}$ by projection along the lines of U will be a local diffeomorphism onto the smooth part of M. Also, $\mathcal{B}$ will only fail to be an embedded manifold due to the nonlocal intersection of points corresponding to distant parts of M. Hence, if there are no such nonlocal intersections, $\mathcal{B}$ is an embedded submanifold.

To establish the smoothness, we consider a "radial flow" which is a type of "backward flow" of the grassfire/level–set flow. However, neither the flow nor its level sets are smooth but rather they are "stratified". While the radial flow can be defined locally on one side of M, to define the flow globally, we introduce the "double of M" on which is defined a "normal bundle" to M. We then use the global radial flow defined on the normal bundle to prove that the skeletal set M has a "tubular neighborhood" (Theorem 5.1). The boundary $\mathcal{B}$ is obtained from the boundary of the tubular neighborhood by flowing out by the radial flow.

The radial flow is an important part of the total geometric structure and reveals the role of the three conditions. The first two conditions allow us to control the local behavior of the radial flow, ensuring that singularities do not develop from smooth points (as in Fig. 2), nor further singularities from singular nor edge points. These conditions are necessary to ensure the level sets of the flow project diffeomorphically along the direction of U onto the smooth points of M. These conditions are shown to be necessary in the Blum case.

While the first two conditions are open conditions and hence robust, the third compatibility condition is not and reveals an essential feature about the level hypersurfaces of the flow. For any time $t < 1$ the level hypersurfaces are singular at points coming from the singular points of M (including edge points). The compatibility condition ensures that only at $t = 1$ when the flow reaches the boundary do the singularities simultaneously disappear so the boundary becomes weakly C^1 at the points corresponding to singular points of M.

In the second part of this paper we shall determine the geometric properties of the boundary in the Blum case. Earlier work sought to relate the differential geometry of the boundary with that of the medial axis modified by differential properties of r. We shall see that a more direct approach follows from analyzing the evolution of these radial and edge shape operators under the radial flow, yielding simple expressions for the differential geometric shape operator on the boundary. As well this allows us to determine the effect of "distorting" diffeomorphisms of the medial structure on the associated boundary via its modification of these operators.

This author would like to thank Stephen Pizer, his coworkers including Sarang Joshi, and students Paul Yushkevich, Andrew Thall, and Tom Fletcher, whose work concerning the geometric and smoothness properties of boundary in terms of the differential geometry of the medial axis in the Blum case, raised questions for more general "medial structures", leading to the methods and results the author introduces here.

CONTENTS

1. Skeletal Structures

Skeletal Sets. We begin by defining exactly what we mean by "skeletal structures". These will include the Blum medial axes in the generic case (as well as many non-generic cases). We consider Whitney stratified sets satisfying extra conditions. We recall a Whitney stratified set M may be represented as a union of disjoint smooth strata M_α satisying the axiom of the frontier: if $M_\beta \cap \bar{M}_\alpha \neq \emptyset$, then $M_\beta \subset \bar{M}_\alpha$; and Whitney's conditions a) and b) (which involve limiting properties of tangent planes and secant lines). For example, for generic boundaries,the Blum medial axis is a Whitney stratified set. This follows from basic properties of Whitney stratified sets (see e. g. [M1] or[Gi]) together with results of Mather [M2] on the distance to the boundary function.

We let M_{reg} denote the points in the top dimensional strata (this is the dimension n of M and these points are the "smooth points"of M). Also, we let M_{sing} denote the union of the remaining strata, and ∂M denote the subset of M_{sing} consisting of points of M at which M is locally an n manifold with boundary, with the points being boundary points. We refer instead to these points as *edge points* to not confuse them with the "boundary associated to M", that we will shortly define. The points of closure of ∂M will be called *edge closure points.* The closure of ∂M will be denoted $\overline{\partial M}$. An example of an edge closure point is a "fin creation point"for the Blum medial axis (see Fig. 6).

Three important consequences of M being a Whitney stratified set are: i) for $x_0 \in M_{sing}$ and $\bar{B}_\varepsilon(x_0)$ a closed ball of radius ε about x_0 for sufficiently small $\varepsilon > 0$, the pair $(\bar{B}_\varepsilon(x_0), M \cap \bar{B}_\varepsilon(x_0))$ is homeomorphic to the cone on $(S_\varepsilon(x_0), M \cap S_\varepsilon(x_0))$, for $S_\varepsilon(x_0)$ the sphere of radius ε (also $M \cap S_\varepsilon(x_0)$ is again a Whitney stratified set and is called the *link of M at x_0*); ii) the local topological structure of M in a sufficiently small neighborhood of x_0 is independent of points x_0 in a given connected component of a stratum M_α of M; and iii) M can be triangulated (see e.g. Goresky [Go] or Verona [V]).

As a consequence, the connected components of $\bar{B}_\varepsilon(x_0)\backslash M$ are well–defined and will be called *the complementary local components for* x_0. Also, the connected components of $(\bar{B}_\varepsilon(x_0) \cap M_{reg})$ are also well–defined. We refer to these as the *neighboring local components of* x_0. By adding the complement of M and refining the strata of M into connected components, we obtain a Whitney stratification of $\mathbb{R}^{n+1}$. Then, in a sufficiently small open neighborhood W of x_0, the boundary of a complementary local component C_i for x_0 will locally be a union of neighboring local components of x_0. We denote this by ∂C_i. We will assume from now on that we have a Whitney stratification with these properties.

If $T = \lim T_{x_i} M$ exists for $x_i \in M_{reg}$ and $\lim x_i = x_0$, then we refer to T as a *limiting tangent space at* x_0.

Definition 1.1. An n–dimensional compact Whitney stratified set $M \subset \mathbb{R}^{n+1}$ is a *skeletal set* if

1. For each local neighboring component M_α of $x_0 \in M_\beta$, there is a unique limiting tangent space $T_{\alpha\, x_0} M$ from sequences of points in M_α (by properties of Whitney stratified sets $T_{x_0} M_\beta \subset T_{\alpha\, x_0} M$),
2. locally in a neighborhood of a singular point x_0, M may be expressed as a union of (smooth) n–manifolds with boundaries and corners M_j, where two such intersect only on boundary facets (faces, edges etc.).

(3) if $x_0 \in \overline{\partial M}$ then those M_j in (2) meeting ∂M meet it in an $n-1$ dimensional facet.

We refer to the M_j as *local manifold components for* x_0. If M_j meets ∂M in an $n-1$ dimensional facet, we call it an *local edge manifold component for* x_0.

Example 1.2. In $\mathbb{R}^2$, a skeletal set which exhibits the generic properties for the Blum medial axis only has simple branching as shown in Fig. 3. A general skeletal set M in $\mathbb{R}^2$ can consist of any finite collection of smooth curve segments which only meet at their end points, as in Fig. 4. Two consecutive curve segments meeting at a branch point separate off a complementary component C_i.

In R^3, models for the generic local behavior for Blum medial axes at nonsmooth points are given in Fig. 6. To illustrate the terminology we consider a fin creation point x_0 as in c) of fig. 6 . There are 2 local complementary components, one above and one below M. The local components of x_0 are decomposed into 5 local manifold components, 4 curved retangular components and a curved triangular component, which is the only edge manifold component for x_0. Possible more general skeletal sets are given in Fig. 7.

For a general skeletal set $M \subset \mathbb{R}^3$, the link of a singular point x_0 is a one dimensional Whitney stratified set in a two sphere, $L \subset S^2$. L consists of a finite number of smooth curve segments which only meet at the ends, i.e. it is a skeletal set in S^2.

The connected components C_i' of $S^2 \backslash L$ correspond to the local complementary components C_i of x_0.

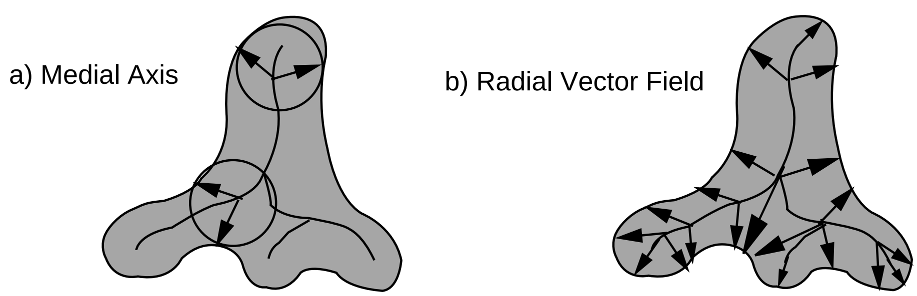

FIGURE 3. Blum Medial axis in $\mathbb{R}^2$ and associated Radial Vector Field

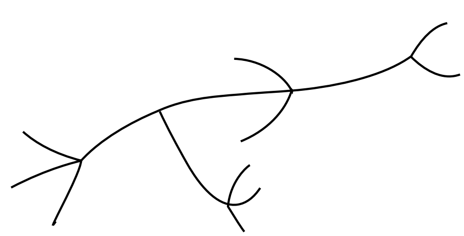

FIGURE 4. Example of General Skeletal Set in $\mathbb{R}^2$

Smoothness of Vector Fields and Functions on Skeletal Sets. To define smooth mappings and vector fields on a skeletal set M, we have to vary somewhat from standard definitions. Usually smooth mappings or vector fields on a (Whitney) stratified sets are obtained as restrictions of the corresponding smooth objects defined on the ambient space. This will not be the case here. To define smoothness of a mapping, resp. vector field, $f : M \to N$ from a skeletal set $M \subset \mathbb{R}^{n+1}$ to a manifold N ($N = \mathbb{R}^{n+1}$ for a vector field f), we require that at regular points of M, f is smooth in the usual sense. At non-edge singular points x_0, we require for each neighboring local component M_α that $f|M_\alpha$ is smooth as a function on manifold with boundary and corners; so in particular M_α can be extended to a smooth manifold in a neighborhood of x_0 and f can also be extended to be smooth.

Lastly, at edge points we must deviate from the standard definition of smoothness on a manifold with boundary to be able to include examples such as the Blum medial axis.

Definition 1.3. An *edge coordinate parametrization* at an edge point $x_0 \in \partial M$ consists of an open neighborhood W of x_0 in M, an open neighborhood $\tilde{W}$ of 0 in $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$ and a differentiable homeomorphism $\phi : \tilde{W} \to W$ such that: both $\phi|\{(x_1, \ldots, x_n) \in \tilde{W} : x_n > 0\}$ and $\phi|(\tilde{W} \cap \mathbb{R}^{n-1})$ are diffeomorphisms.

An *edge coordinate parametrization* for a edge manifold component M_j of *an edge closure point* x_0 consists of an edge coordinate parametrization for an extension of M_j to a smooth manifold with boundary containing x_0 in its boundary.

Then, by smoothness of a mapping or vector field on a skeletal set M at an edge (or edge closure) point x_0 we will mean that the composition with an edge parametrization is smooth. As usual the edge parametrization must be compatible with coordinate charts on M_{reg} (and for edge parametrizations at other edge points). It follows that a vector field or function on M which is smooth at an edge point when viewed as being on a manifold with boundary, then they are also smooth for the edge parametrization (but not conversely, see Example 1.4).

For a multivalued vector field U on M, by a *smooth value of* U at a point $x_0 \in M_{reg}$, we mean a neighborhood W of x_0 in M and a choice of values of U at points of W which together form a smooth vector field on W. This extends to a local component M_α of a point $x_0 \in M_{sing}$. We use analogous notation for multivalued functions such as the radial function.

Example 1.4 (Smoothness of the radial vector field of the medial axis)**.** The Blum medial axis M of an object/region $\Omega \subset \mathbb{R}^{n+1}$ with smooth boundary $\mathcal{B}$ is the Maxwell set (or conflict set) of the family of distance functions on the boundary $f(x, y) = \|x - y\| : \mathcal{B} \times (\mathbb{R}^{n+1} \backslash \mathcal{B}) \to \mathbb{R}$ viewed as a function of x with parameters y (see [M2], [BGG], [Brz] and more recently [Gb]). The Maxwell set consists of parameter values y for which f, as a function of x, has two local minima with the smallest local minimum value.

Along with the distance to the boundary function is the canonical vector field $\tilde{U}(x, y) = (0, x - y)$ defined on $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$. If S denotes the set of critical points with common minimum value lying above the Maxwell set, then the restriction of $\tilde{U}$ to S projects to the multivalued Blum radial vector field U on M.

In the generic case, the regular points of M correspond to "$A_1^2 = A_1A_1$"points of f; the branch points, to "$A_1^3 = A_1A_1A_1$"points; the edge points, to "A_3"points; fin creation points, to "A_1A_3"points; and "6- junction"points to "A_1^4"points. For

$\dim M = n \leq 2$, these describe all of the generic behavior. In S the corresponding sets are the transverse intersection of some combination of A_1 and A_3 critical sets which are smooth manifolds. The projections to $\mathbb{R}^{n+1}$ of the A_1^2, A_1^3, and A_1^4 sets are diffeomorphisms onto the smooth, resp. branch points of M, and the composition of U with the projection is $\tilde{U}$, so U smooth on M. For A_1A_3 points, on the local components other than the edge component, the two singular behaviors are A_1 and A_3 behavior. For the portion of the boundary corresponding to A_1 singularities, in a neighborhood in $\mathbb{R}^{n+1}$ of the fin creation point, there are unique minimum vectors to this portion of the boundary which vary smoothly with the points. The radial vectors on this side are those coming from points on the medial axis. Hence, the radial vector field on the one side for the non–edge local components is smooth. As the radial vector pointing in the other direction is the orthogonal reflection of this vector field through the tangent spaces of M, we conclude the radial vector fields on both sides are smooth.

There remain edge points (A_3) and edge closure points (edge components for A_1A_3–points). Because the edge component for a fin creation point continues smoothly through the fin creation point (although not on the medial axis), this case reduces to the case of an edge point. For an edge point (A_3), we consider the projection to $\mathbb{R}^{n+1}$ of the A_3 singular set. Unlike smooth and branch points, this projection is not a local diffeomorphism. In the generic case for A_3 points, f is given, after change of coordinates in y and parametrized change of coordinates in x, by the standard model

$$f(x, y) = x_1^4 + y_1 x_1^2 + y_2 x_1 + \sum_{i=2}^{n} \pm x_i^2 .$$

For this model equation, $S = \{(x, y) : y_2 = 0, y_1 = -2x_1^2, x_i = 0, i \geq 2\}$ which maps by a $2-1$ fold map onto the Maxwell set M. Then, an edge coordinate parametrization is given by the projection restricted to half of S. Specifically, it is defined by $(t_1, \dots, t_n)$ with $t_1 \geq 0$ mapping into S by $(t_1, \dots, t_n) \mapsto (t_1, 0, \dots 0, -2t_1^2, t_2, \dots, t_n)$ which then projects onto M (see Fig. 5) The Blum (multivalued) radial vector field U is the projection of $\tilde{U}$ on S, and it is smooth when composed with the edge parametrization (as the projection on one half of S is obtained by composing the diffeomorphism sending $x_1 \mapsto -x_1$ with the projection on the other half). For A_1A_3 (fin creation) points, the edge parametrization for A_3 is intersected with the A_1 submanifold so U composed with the edge parametrization is again smooth.

Hence, for edge points U is also smooth on M using edge coordinates (hence, so also is the radial function $r = \|U\|$).

Radial Vector Fields on Skeletal Sets.

Definition 1.5. Given an n–dimensional skeletal set $M \subset \mathbb{R}^{n+1}$, a *radial vector field* U on M is a nowhere zero multivalued vector field satisfying the following conditions:

(1) (Behavior at smooth points) For each smooth point $x_0 \in M_{reg}$, there are two values of U which are on opposite sides of $T_{x_0}M$ (i.e. their dot products with a normal vector are nonzero with opposite signs). Moreover, on a neighborhood of a point of M_{reg}, the values of U corresponding to one side form a smooth vector field.

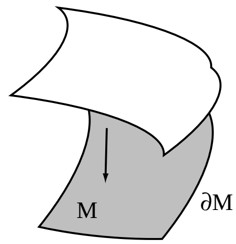

FIGURE 5. Edge parametrization for neighborhood of an edge point on a skeletal set

(2) (Behavior at non-edge singular points) Let $x_0 \notin \partial M$, be a singular point, with M_α a local component of x_0. Then, both smooth values of U on M_α extend smoothly to values $U(x_0)$ on the stratum of x_0. If M_α does not intersect ∂M in a neighborhood of x_0, then $U(x_0) \notin T_{\alpha\, x_0} M$. Conversely, to each value of U at a point $x_0 \in M_{sing}$, there corresponds a local complementary component C_i of M at x_0 such that the value $U(x_0)$ *locally points into* C_i in the following sense. The value $U(x_0)$ extends smoothly to values $U(x)$ on the local components of M for x_0 in ∂C_i. For a neighborhood W of x_0 and an $\varepsilon > 0$, $x + tU(x) \in C_i$ for $0 < t < \varepsilon$ and for $x \in (W \cap M)$.

(3) (Tangency behavior at edge points) At points $x_0 \in \partial M$, there is a unique value for U which is tangent to the stratum of M_{reg} containing x_0 in the closure and points away from M.

Because of the properties of the radial vector field, we shall see that the singular points of M are naturally subdivided into edge points, edge closure points and the remaining singular points. The tangency of the radial vector field at edge points contrasts with the transverse behavior at non-edge closure points. At edge closure points such as fin creation points, there will be a mixture of both conditions.

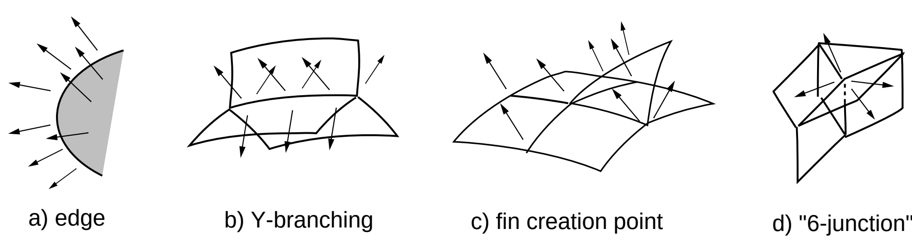

FIGURE 6. Local generic structure for Blum Medial axes in $\mathbb{R}^3$ and the associated Radial Vector Fields

For a radial vector field U, we may represent $U = r \cdot U_1$, for a positive multivalued function r, and a multivalued unit vector field U_1 on M. These satisfy analogous properties to U, namely U_1 is a radial vector field on M, and on small neighborhoods of points in M_{reg}, there are two smooth nonvanishing choices for values of r, and r extends smoothly to points $x_0 \in M_{sing}$ along local components of M_{reg} containing x_0 in the closure.

Remark 1.6. For a radial vector field U to be a *Blum radial vector field* it satisfies additional properties: at each smooth point x_0, the two values $U^{(1)}$ and $U^{(2)}$ must

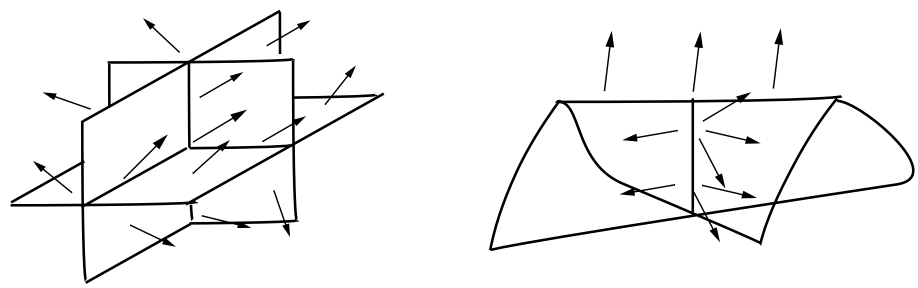

FIGURE 7. Possible nongeneric skeletal structures in $\mathbb{R}^3$ and associated Radial Vector Fields

satisfy $\|U^{(1)}\| = \|U^{(2)}\|$ and $U^{(1)} - U^{(2)}$ is orthogonal to $T_{x_0}M$. However, neither of these properties are necessary for smoothness of the boundary nor to determine the geometry. Instead, we shall see the key relevant Blum condition is given by a compatibility condition via the compatibility 1–form (see §§2 and 6).

We introduce additional "initial conditions" necessary for a radial vector field on a skeletal set to ensure that near singular points the radial flow we shall define is initially well behaved. Some such conditions are necessary as illustrated in Example 1.8.

Definition 1.7. A radial vector field U on a skeletal set M satisfies the *local initial conditions* if it satisfies the following :

(1) (*local separation property*) For a local complementary component C_i of a non–edge point $x_0 \notin \partial M$, let $\partial C_i = \cup M_i$ denoting the local decomposition of ∂C_i into closed (in W) n-manifolds with boundaries and corners. Then the set $X = \{x + tU(x) : x \in \cup_i \partial M_i, 0 \leq t \leq \varepsilon\}$ is an embedded Whitney stratified set such that distinct $\text{int}(M_i)$ and $\text{int}(M_j)$ lie in separate connected components of the complement of $\bar{C}_i \backslash X$.

(2) (*local edge property*): for each edge (closure) point $x_0 \in \overline{\partial M}$, there is a neighborhood W of x_0 in M and an $\varepsilon > 0$ so that for each smooth value of U, the radial flow $\psi(x, t) = x + t \cdot U$ is one–one on $W \times [0, \varepsilon]$.

Example 1.8. The tangency behavior of radial vector fields at edge points $x_0 \in \partial M$ require the special "local edge property" which is unnecessary for other singular points. It is necessary because there are manifolds such as the graph M of the function $f(x) = \exp(\frac{-1}{x^2}) \cdot \cos(\frac{1}{x})$ for $x < 0$ and $f(0) = 0$. The tangent line to M at 0 is the x–axis. However, there are radial vector fields on M whose angle decreases faster than $\exp(\frac{-1}{x^2})$ so that on no neighborhood of 0 is the local edge property satisfied (see Fig. 8 a).

Also, the "local separation property" for singular points is illustrated in Fig. 8 b). That it is also necessary is illustrated in Fig. 8 c), where we exhibit a vector field which satisfies all of the properties of a radial vector field except for the local separation property. The associated boundary can then have unavoidable singularities.

We shall see that the conclusion of Theorem 2.5 already follows for a skeletal structure (M, U) consisting of radial vector field U on a skeletal set M of dimension ≤ 2 and satisfying the local initial conditions. However, for higher dimensional skeletal sets M subtle points from geometric topology prevent us from deducing the

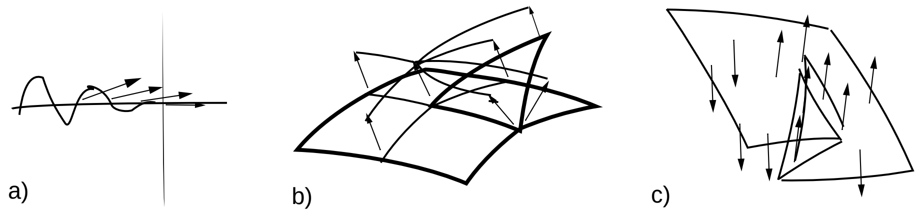

FIGURE 8. a) Failure of Local Edge Property b) Illustration of the Local Separation Property c) Failure of Local Separation property

full theorem without further restictions on the local structure of M. We address this issue next.

Abstract Boundaries and the Topology of Local Complementary Components. Suppose C_i is a local complementary component of a singular point $x_0 \in M_{sing}$. We may then express the local boundary of C_i in a sufficiently small open neighborhood W of x_0 as a union of n–manifolds with boundaries and corners $\{M_i, i = 1, \dots, k\}$ (which are closed in W). We introduce an "abstract version of the boundary"of C_i. It consists of a copy of M_i for each smooth value of U on M_i pointing into C_i (there are, thus, at most two copies of a given M_i). We then make identifications on facets $M_i \cap M_j$ of boundaries of copies of M_i and M_j if the corresponding smooth values for these copies agree on $M_i \cap M_j$. We denote the resulting space by $\partial\tilde{C}_i$.

Definition 1.9. The space $\partial\tilde{C}_i$ just constructed will be called the *abstract boundary of* C_i.

For example, b) of Fig. 11 shows the abstract boundary for the upper complementary component of a fin creation point. In the case $x_0 \in \partial M$, the abstract boundary of the single complementary component is a "double"of the local manifold with boundary near x_0 in the usual sense of differential topology, (see e.g. [Mu]).

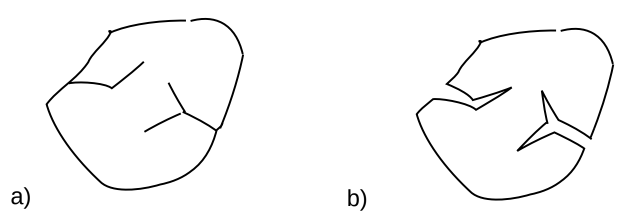

FIGURE 9. a) Link for a boundary of a complementary component of a singular point b) Link of the associated abstract boundary for the same complementary component

Example 1.10 (Abstract Boundaries in $\mathbb{R}^2$ and $\mathbb{R}^3$). For a general skeletal set $M \subset \mathbb{R}^2$, which consists of a finite collection of smooth curve segments only meeting at their end points, two curve segments meeting at a branch point separate off a complementary component C_i. As only one smooth value on a curve segment can locally point into C_i, the abstract boundary agrees with ∂C_i.

By our earlier discussion, for a general skeletal set $M \subset \mathbb{R}^3$, the link L of a singular point x_0 is a one dimensional Whitney stratified set in a two sphere, $L \subset S^2$; and the connected components C_i' of $S^2 \backslash L$ correspond to the local complementary components C_i of x_0. Then, the properties of the abstract boundaries in terms of C_i' are given by Lemma 1.12. It follows that for skeletal sets with radial vector fields in $\mathbb{R}^3$, the abstract boundaries will be homeomorphic to 2–disks.

More generally for a skeletal set which supports a radial vector field satisfying the local initial conditions, the complementary components are contractible.

Lemma 1.11. *Let $M \subset \mathbb{R}^{n+1}$ be a skeletal set with radial vector field U satisfying the local initial conditions. Both the complementary component C_i of any point in M and the associated complementary component C_i' are contractible.*

The proof of this lemma will be given in §5. We deduce a consequence for skeletal sets in $\mathbb{R}^3$.

Lemma 1.12. *Let $M \subset \mathbb{R}^3$ a skeletal set with radial vector field U satisfying the local initial conditions. Let C_i be a complementary component of $x_0 \in M$, with associated complementary component C_i', then the associated abstract boundary of C_i is homeomorphic to a 2–disk. If instead some complementary component C_i' of M is not simply connected, then M does not support a radial vector field satisfying the local initial conditions.*

Proof. By Lemma 1.11 C_i' is contractible. However, its closure $\bar{C}_i'$ may not be contractible. Also, its topological boundary will consist of: the piecewise smooth boundary of $\bar{C}_i'$ along with a finite number of tree structures extending into the interior of $\bar{C}_i'$ (see, for example, a) of Fig. 9). As one follows the boundary $\bar{C}_i'$, choosing at branch points so the inward pointing vector field varies continuously, we follow the entire boundary (except branch points) exactly once. Hence, the boundary of $\bar{C}_i'$ is the image of an S^1, and hence, $\bar{C}_i'$ is the image of a 2–disk identified at a finite number of points on the boundary. The construction of the abstract boundary will first remove the intersection points on the boundary. Second, for each segment of tree structure inward from the boundary, it will cut it replacing it by a pair of unidentified edges (see b) of Fig. 9). The resulting object is still a two disk with boundary. After a finite number of such cuts, we obtain the link of the abstract boundary. Thus, the abstract boundary is the cone on this link and hence homeomorphic to a 2–disk.

For the second part, by Lemma 1.11 if the skeletal set supports a radial vector field, then any C_i' is contractible. □

We include this as a restriction on M in higher dimensions as part of our definition of a skeletal structure.

Definition 1.13. A *Skeletal Structure* (M, U) in $\mathbb{R}^{n+1}$ consists of an n–dimensional skeletal set $M \subset \mathbb{R}^{n+1}$ and a radial vector field U on M satisfying the local initial conditions, such that all abstract boundaries of local complementary components are homeomorphic to n–disks.

Remark 1.14. We note that for regions with piecewise smooth boundary, the Blum medial axis ends at "corners of the boundary" and the radius function r becomes 0 at such points. We are specifically excluding from consideration behavior at such points as our radial vector field is always nonzero.

Definition 1.15. Given a skeletal structure (M, U), we define the *associated boundary*

$$\mathcal{B} \stackrel{\text{def}}{=} \{x + U(x) : x \in M\},$$

where the definition includes all values of $U(x)$ for a given x.

Example 1.16. By our discussion in Example 1.4 if $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$ is a region with a generic smooth boundary $\mathcal{B}$, the Blum medial axis M is a skeletal set. Moreover, the associated radial vector field U consisting of vectors from points in M to the associated points of tangency to $\mathcal{B}$ is a radial vector field satisfying the local initial conditions. Thus, in the generic case, (M, U) is a skeletal structure. Furthermore, the original boundary $\mathcal{B}$ is the boundary of the skeletal structure obtained from the Blum medial axis.

On the other hand, there are also many circumstances where a nongeneric medial axis with its radial vector field is a skeletal structure.

2. Shape Operators and Smoothness of Associated Boundaries

We are now ready to introduce the shape operators associated to skeletal structures which will be fundamental for understanding the structure of the boundary.

Radial Shape Operators and Principal Radial Curvatures. Given a skeletal structure (M, U) in $\mathbb{R}^{n+1}$, we define for a regular point x_0 and each smooth value of U defined in a neighborhood of x_0, with associated unit vector field U_1, a *radial shape operator*

$$S_{rad}(v) \quad = \quad -\text{proj}_U(\frac{\partial U_1}{\partial v})$$

for $v \in T_{x_0}M$. Here proj_U denotes projection onto $T_{x_0}M$ along U (which in general is not orthogonal to $T_{x_0}M$). We observe that $S_{rad} : T_{x_0}M \to T_{x_0}M$ is linear. However, because U_1 is not necessarily normal and the projection is not orthogonal, it does not follow that S_{rad} is self–adjoint as is the case for the usual differential geometric shape operator. However, this operator does measure how U bends relative to M.

First, we choose a basis $\{v_1, \dots, v_n\}$ for $T_{x_0}M$ and for each i represent

$$\frac{\partial U_1}{\partial v_i} \quad = \quad a_i \cdot U_1 - \sum_{j=1}^{n} s_{ji} v_j \tag{2.1}$$

We write this and other equations in vector form. We let $\mathbf{v}$ denote the column vector with i–th entry v_i, $A_{\mathbf{v}}$ with i–th entry a_i, $\frac{\partial U_1}{\partial \mathbf{v}}$ with i–th entry $\frac{\partial U_1}{\partial v_i}$. Also, $S_{\mathbf{v}}$ is the matrix with ij–th entry s_{ij} and is a matrix representation for S_{rad} with respect to the basis $\{v_1, \dots, v_n\}$. Then, (2.1) can be written in vector form by

$$\frac{\partial U_1}{\partial \mathbf{v}} \quad = \quad A_{\mathbf{v}} \cdot U_1 - S_{\mathbf{v}}^T \cdot \mathbf{v} \tag{2.2}$$

In this equation we interpret $A_{\mathbf{v}} \cdot U_1$ as the column vector with i–th entry the vector $a_i \cdot U_1$; while $S_{\mathbf{v}}^T \cdot \mathbf{v}$ denotes the column matrix obtained by matrix multiplication of the scalars in $S_{\mathbf{v}}^T$ (the transpose of $S_{\mathbf{v}}$) times the vectors in $\mathbf{v}$.

Although U_1 not being orthogonal to M prevents $S_{\mathbf{v}}$ from being self-adjoint, $\|U_1\| = 1$ implies $\frac{\partial U_1}{\partial v_i} \cdot U_1 = 0$. Applying dot product with U_1 to each entry in

(2.2) allows us to solve

$$A_{\mathbf{v}} \quad = \quad S_{\mathbf{v}}^T \cdot (\mathbf{v} \cdot U_1) \tag{2.3}$$

where $\mathbf{v} \cdot U_1$ denotes the column vector with i–th entry $v_i \cdot U_1$

Definition 2.1. For a point x_0 and a given smooth value of U, we call the eigenvalues of the associated operator S_{rad} the *principal radial curvatures* at x_0, and denote them by $\kappa_{r\,i}$.

Remark 2.2. We emphasize that because there are two smooth values of U at smooth points ,we obtain two shape operators at each point. Moreover, near a non–edge point $x_0 \in M_{sing}$, for each local smooth component of M_{reg} for x_0, each smooth value of U will extend smoothly to x_0. Thus, to each value of U and each local component, such a shape operator will be defined at x_0. Hence, any statement involving the shape operator will involve all of these for each point.

Edge Shape Operators on ∂M. Next, we define an Edge–shape operator at points of ∂M which will measure how U_1 bends relative to ∂M. Given a point $x_0 \in \partial M$ and a smooth value U at x_0, we let $\mathbf{n}$ be the unit normal vector field to M in a neighborhood of x_0. Then, we define the *Edge–shape operator* by

$$S_E(v) \quad = \quad -\mathrm{proj}'(\frac{\partial U_1}{\partial v})$$

for $v \in T_{x_0}M$. Here proj$'$ denotes projection onto $T_{x_0}\partial M \oplus < \mathbf{n} >$ along U_1.

Alternately if x_0 is an edge closure point with M_j an edge manifold component of x_0, we can define an edge shape operator using the smooth value of U defined on M_j. To simplify the discussion, we concentrate on the case of an edge point with the appropriate substitution of notation to be understood for an edge closure point.

Given a basis $\{v_1, \dots, v_{n-1}\}$ of $T_{x_0}\partial M$, we also choose a vector v_n in the edge coordinate system at x_0 so that $\{v_1, \dots, v_{n-1}, v_n\}$ is a basis $T_{x_0}M$ in the edge coordinate system and so that v_n maps under the edge parametrization map to $c \cdot U_1(x_0)$ where $c \geq 0$ (the specific value of c is immaterial). Then, we can compute a matrix representation $S_{E\,\mathbf{v}}$ for S_E in a manner analogous to (2.2). Let $\mathbf{n}$ be a unit normal vector field to M on a neighborhood W of x_0.

$$\frac{\partial U_1}{\partial v_i} \quad = \quad a_i \cdot U_1 - c_i \cdot \mathbf{n} - \sum_{j=1}^{n-1} b_{ji} v_j \tag{2.4}$$

As in (2.2), this equation can be written in vector form

$$\begin{aligned} \frac{\partial U_1}{\partial \mathbf{v}} \quad &= \quad A_U \cdot U_1 - C_U \cdot \mathbf{n} - B_{U\,\mathbf{v}} \cdot \tilde{\mathbf{v}} \\ &= A_U \cdot U_1 - \begin{pmatrix} B_{U\,\mathbf{v}} & C_U \end{pmatrix} \cdot \begin{pmatrix} \tilde{\mathbf{v}} \\ \mathbf{n} \end{pmatrix} \end{aligned} \tag{2.5}$$

where the vectors A_U and C_U are again n–dimensional column vectors, $B_{U\,\mathbf{v}}$ is an $n \times (n-1)$–matrix, and $\tilde{\mathbf{v}}$ is the $n-1$ dimensional vector with entries $v_1, \dots, v_{n-1}$. Then, $S_{E\,\mathbf{v}} = (B_{U\,\mathbf{v}}\ C_U)^T$.

To define the principal edge curvatures of M at x_0, we use generalized eigenvalues of $S_{E\,\mathbf{v}}$. Recall that the generalized eigenvalues of an ordered pair (A, B) of $n \times n$–matrices consists of λ such that $A - \lambda \cdot B$ is singular. We let $I_{n-1,1}$ denote the $n \times n$–diagonal matrix with 1's in the first $n-1$ diagonal positions and 0 otherwise.

We call the generalized eigenvalues of $(S_{E\,\mathbf{v}}, I_{n-1,1})$ the *principal edge curvatures of* M and denote them by $\{\kappa_{E\,i}\}$.

Remark 2.3. The matrix representation $S_{E\,\mathbf{v}}$ is unusual in that we change the basis in the source to the target by replacing v_n by $\mathbf{n}$. We shall see that this is exactly what is needed. Also, we note that had we multiplied v_n by a nonzero constant, this would only change the last column of $S_{E\,\mathbf{v}}$. However, this would not alter the generalized eigenvalues of $(S_{E\,\mathbf{v}}, I_{n-1,1})$. Finally, for the edge condition we are only really interested in the real (positive) generalized eigenvalues.

Compatibility 1*-forms.* We now have defined two of the three invariants needed for the conditions introduced at the beginning of this paper. For the third condition, we define the *compatibility* 1*-form*. Given a smooth value for U, (possibly at a point of M_{sing}), we write $U = r \cdot U_1$ for a unit vector field U_1 and define $\omega_U(v) = v \cdot U_1$. Then, the *compatibility* 1*-form* $\eta_U \stackrel{\text{def}}{=} \omega_U + dr$. This is a multivalued 1–form.

Remark 2.4. The compatibility condition requires that η_U vanish at all points of M_{sing}. We shall see in Lemma 6.1 that the vanishing of η_U at x_0 implies that $U(x_0)$ is orthogonal to the tangent space of the associated boundary $\mathcal{B}$ at the corresponding point.

Three Conditions Implying the Smoothness of the Boundary. Then the main result of this first part of the paper is the following theorem.

Theorem 2.5. *Let (M, U) be a skeletal structure which satisfies: the Principal Curvature Condition, Edge Condition, and Compatibility Condition. Then,*

1. *the associated boundary $\mathcal{B}$ is an immersed topological manifold which is smooth at all points except those corresponding to points of M_{sing}.*
2. *At points corresponding to points of M_{sing}, it is weakly C^1 (this implies that it is C^1 on the points which are in the images of strata of codimension* 1*).*
3. *At smooth points, the projection along the lines of U will locally map $\mathcal{B}$ diffeomorphically onto the smooth part of M.*
4. *Also, if there are no nonlocal intersections, $\mathcal{B}$ will be an embedded manifold.*

Remark 2.6. In all that we do, we assume that the strata of M and U are C^∞. However, if we have weaker differentiability assumptions of C^k for $k \geq 1$, then corresponding C^k statements hold for $\mathcal{B}$.

Example 2.7. In [D2] we shall consider in much greater detail the conditions for (M, U) a skeletal structure in $\mathbb{R}^2$ or $\mathbb{R}^3$.

For now we just remark on the simplest case of a 1–dimensional skeletal structure (M, U) in $\mathbb{R}^2$. If $\gamma(s)$ is a local parametrization of one of the smooth components of M, then write

$$\frac{\partial U_1}{\partial s} \quad = \quad a \cdot U_1 - \kappa_r \cdot \gamma'(s) \tag{2.6}$$

The radial shape operator is multiplication by the principal radial curvature κ_r, and the radial curvature condition becomes (for each side of M):

$$r \;<\; \frac{1}{\kappa_r} \quad \text{if } \kappa_r > 0, \text{ and no condition otherwise.}$$

At a singular point x_0, this condition must hold for each side of each local component of x_0. We constrast this with the necessary condition at smooth points obtained for the Blum case by Pizer and Yushkevich which makes use of the differential geometry of the medial axis (see [P3, §2])

$$r'^2 + \frac{r}{|r_{axis}|} \cdot \sqrt{1 - r'^2} - rr'' \quad \leq \quad 1$$

where primes denote derivatives and r_{axis} is the radius of curvature of the medial axis.

3. Radial Flow and Tubular Neighborhood for a Skeletal Structure

Consider a skeletal structure (M, U) satisfying the conditions of Theorem 2.5. As a first step to proving the smoothness of the boundary associated to (M, U), we begin by defining a global radial flow. We recall that one way to view the formation of the medial axis is as the shock set resulting from the Grassfire/level-set flow from the boundary (see e.g. a) of Fig. 10) Kimia et al [KTZ], (and also Siddiqi et al [SB] and [P3] for further discussion). This flow is from points on the boundary along the normals until shocks are encountered. The radial flow we will consider is essentially a "backward flow"along U to relate the skeletal set M with the boundary $\mathcal{B}$. However, this does not give a well-defined flow (or function) since U is multivalued. In addition, we measure distance along U radially from M so the level hypersurfaces do not correspond to those obtained by flowing from the boundary. In fact, the radial hypersurfaces are no longer nonsingular nor are they orthogonal to the lines of flow.

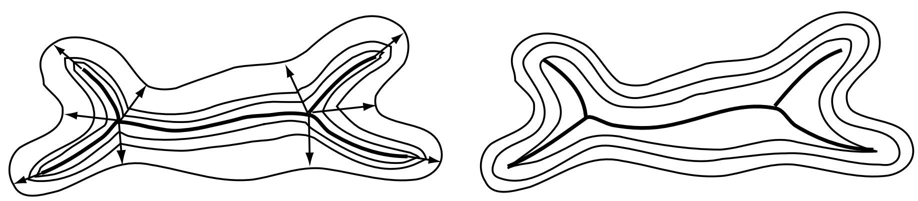

Figure 10. a) Radial Flow b) Grassfire/Level–set Flow in $\mathbb{R}^2$

Nonetheless, we define and investigate the properties of this "radial flow". We can intrinsically define the level sets for each $0 \leq t \leq 1$, defined by $\mathcal{B}_t = \{x + t \cdot U(x) : x \in M\}$ so $\mathcal{B}_0 = M$ and $\mathcal{B}_1 = \mathcal{B}$. Locally if we choose a smooth value of U defined on a neighborhood W of $x_0 \in M$, we can define a local radial flow $\psi(x, t) = x + t \cdot U(x)$ on $W \times I$, which flows through the level hypersurfaces. We cannot use such local radial flows to define a global one on M because the radial vector field U is multivalued on M. We overcome this problem by introducing a form of "normal bundle"for (M, U), except that it is defined on the "double"$\tilde{M}$ of M which we will also define.

The Double and the Normal Bundle of M and the Global Radial Flow. For a closed smooth submanifold, the tubular neighborhood is defined using the normal bundle. A skeletal set does not have a normal bundle in the usual sense. We get around this problem by first introducing the double of M.

Let (M, U) be a skeletal structure. We consider

$$\tilde{M} \;=\; \{(x, U') \in M \times \mathbb{R}^{n+1} : U' \text{ is a value of } U \text{ at x } \}$$

To put a topology on $\tilde{M}$, we describe a system of neighborhoods for each point. If $x_0 \in M_{reg}$, with a value $U(x_0)$, then neighborhoods of $(x_0, U(x_0))$ consists of the intersection of $\tilde{M}$ with $W \times \{U_0\}$ where W is a neighborhood of x_0 in M_{reg} and $\{U_0\}$ denotes the set of values for a continuous extension U_0 of $U(x_0)$ to W. If $x_0 \in M_{sing}$, then $U(x_0)$ points into some complementary component C_i of x_0. Then, a neighborhood system for $(x_0, U(x_0))$ is defined as follows: given a the neighborhood W of x_0, we define a neighborhood as the intersection of $\tilde{M}$ with a set of the form $(W' \cap \partial C_i) \times \{U_0\}$, where $W' \subset W$ is an open neighborhood of x_0 in $\mathbb{R}^{n+1}$ and U_0 is a continuous extension of $U(x_0)$ to $W' \cap \partial C_i$. If $x_1 \in W' \cap \partial C_i$ and $x_1 \in M_{sing}$, then for a neighborhood W_1 of x_1 with $W_1 \subset W'$, the complementary component C_i' for x_1 for the value $U_0(x_1)$ pointing into C_i satisfies $\partial C_i' \subset \partial C_i$, so $(W_1 \cap \partial C_i') \subset (W' \cap \partial C_i)$. If U_1 denotes the restriction of U_0 to $W_1 \cap \partial C_i'$ then $(W_1 \cap \partial C_i') \times \{U_1\} \subset (W' \cap \partial C_i) \times \{U_0\}$. Thus, these neighborhoods give a well defined topology. We will refer to these neighborhoods as *abstract neighborhoods* of points in M. An example is shown in Fig. 11 for the case of a "fin creation point". In general, an abstract neighborhood corresponds to the abstract boundary $\partial\tilde{C}_i$ of a complementary component C_i of a point because the values of U on an abstract boundary of C_i and pointing into C_i are uniquely determined. For an edge point $x_0 \in \partial M$, there is a unique point $(x_0, U(x_0)) \in \tilde{M}$ corresponding to it, and a neighborhood of $(x_0, U(x_0))$ in $\tilde{M}$ is a "double of the local manifold with boundary"in the sense of [Mu].

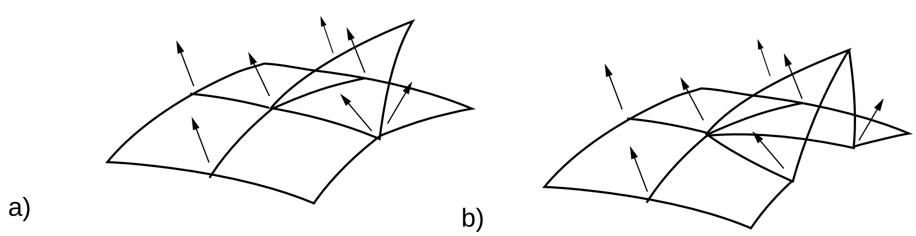

FIGURE 11. a) Neighborhood of a "fin creation point" b) Corresponding Abstract Neighborhood

The natural projection $p : \tilde{M} \to M$, sending $(x_0, U_0) \mapsto x_0$ is continuous and is a k to 1 covering of any connected stratum of M with k values at each point. We can naturally introduce a smooth structure on $\tilde{M}_{reg} = p^{-1}(M_{reg})$, and p is smooth on $\tilde{M}_{reg}$. Moreover, on $\tilde{M}$ we have a canonical line bundle N which at a point (x_0, U_0) is spanned by U_0.

Definition 3.1. For a skeletal structure (M, U), we call $\tilde{M}$ the *double of* M, and N the *normal line bundle to* M (strictly speaking they are the double and normal line bundle for (M, U)).

As usual, $\tilde{M}$ embeds as (and will be identified with) the zero section of N. Also, given an $\varepsilon > 0$ we have the *positive* ε *neighborhood of the zero section* $N_\varepsilon = \{(x_0, tU_0) \in N : 0 \leq t \leq \varepsilon\}$.

Now, for (M, U), with normal line bundle N, we can define the *global radial flow* as a map $\tilde{\psi} : N \to \mathbb{R}^{n+1}$ by $(x_0, tU_0) \mapsto x_0 + tU_0$.

The Tubular Neighborhood of a Skeletal Structure. Now we define an associated tubular neighborhood of a skeletal structure.

Definition 3.2. By a *tubular neighborhood* $|N|$ of a skeletal structure (M, U) in $\mathbb{R}^{n+1}$ we mean there is an $\varepsilon > 0$ so that the global radial flow $\tilde{\psi}|N_\varepsilon$ is a homeomorphism on $N_\varepsilon \backslash \tilde{M}$ with image disjoint from M, and $\tilde{\psi}(N_\varepsilon)$ is a topological neighborhood of M in $\mathbb{R}^{n+1}$.

The tubular neighborhood $|N|$ consists of the level hypersurfaces $\mathcal{B}_t = \{x + tU(x) : x \in M\}$ for $0 \leq t \leq \varepsilon$ and has the property: on $|N|\backslash M$ the radial flow gives a well–defined flow defined for $0 < t < t' \leq \varepsilon$ by $\tilde{\psi}(x, t \cdot U_0, t - t') \mapsto \tilde{\psi}(x, t' \cdot U_0)$. it defines a homeomorphism $\mathcal{B}_t \to \mathcal{B}_{t'}$, which is a smooth diffeomorphism on the points coming from M_{reg} under the radial flow. We shall later use this flow and abuse terminology by also referring to it as the radial flow on $|N|\backslash M$.

We do not require that the level hypersurfaces $\mathcal{B}_t$ be topological manifolds. However, we shall deduce this when we prove existence, as well as proving smoothness of the image off M_{sing}. However, the boundary of the tubular neighborhood will typically be nondifferentiable on the points which are images of M_{sing}. Also, we shall show each $\mathcal{B}_t$ is still a Whitney stratified set.

We shall prove the existence of a tubular neighborhood for a skeletal structure (M, U) in §5.

4. Properties of the Radial Flow

We consider the local behavior of the radial flow on neighborhoods of the various types of points in $\tilde{M}$.

Local Properties of the Radial Flow. We begin by considering $\tilde{\psi}$ in a neighborhood of a regular point $\tilde{x}_0$. It corresponds to a regular point x_0 of M together with a value $U(x_0)$. Thus, we can consider a smooth extenson of U on an open neighborhood W of $x_0 \in M_{reg}$. Then, $\tilde{\psi}$ can be represented by the smooth map $\psi(x, t) = x + t \cdot U(x) : W \times [0, 1] \to \mathbb{R}^{n+1}$ which we refer to as the local radial flow. For fixed t we obtain a map $\psi_t : W \to \mathbb{R}^{n+1}$.

First we compute the derivative of the local radial flow. For $x_0 \in W$, let $\{v_1, \ldots, v_n\}$ be a basis for $T_{x_0}M$. Then, as $U = r \cdot U_1$, we compute

$$\frac{\partial \psi}{\partial v_i} = v_i + t(\frac{\partial r}{\partial v_i} \cdot U_1 + r \cdot \frac{\partial U_1}{\partial v_i})$$

and using (2.1) and $\frac{\partial r}{\partial v_i} = dr(v_i)$

$$= t(dr(v_i) + ra_i) \cdot U_1 + (\sum_{j=1}^{n} (\delta_{ij} - tr \cdot s_{ji}) v_j \tag{4.1}$$

We can rewrite these equations for $i = 1, \ldots, n$ in vector form as

$$\frac{\partial \psi}{\partial \mathbf{v}} \;=\; t(dr(\mathbf{v}) + r \cdot A_{\mathbf{v}}) \cdot U_1 + (I - tr \cdot S_{\mathbf{v}})^T \cdot \mathbf{v} \tag{4.2}$$

where $\dfrac{\partial \psi}{\partial \mathbf{v}}$ and $dr(\mathbf{v})$ are column vectors with i–th entries $\dfrac{\partial \psi}{\partial v_i}$, resp. $dr(v_i)$, and I is the $n \times n$ identity matrix. We also trivially note

$$\frac{\partial \psi}{\partial t} \; = \; r \cdot U_1 \tag{4.3}$$

Local Nonsingularity of the Flow from Smooth Points. Using the preceding, we can deduce the local nonsingularity of ψ.

Proposition 4.1. *Suppose that for the smooth value of U in a neighborhood W of $x_0 \in M_{reg}$, the associated radial operator S_{rad} satisfies*

$$r \; < \; \min\{\frac{1}{\kappa_{r\,i}}\} \quad \textit{for all positive radial principal curvatures } \kappa_{r\,i} \tag{4.4}$$

Then,

(1) $\psi : W \times \mathbb{R} \to \mathbb{R}^{n+1}$ *is a local diffeomorphism at* $(x_0, 0)$
(2) $\psi_t : W \to \mathbb{R}^{n+1}$ *is a local embedding at* x_0 *for any* $0 \leq t \leq 1$.
(3) $\psi_t(W)$ *is transverse to the line spanned by* U *for each* $0 \leq t \leq 1$.

Remark 4.2. Then, 2) guarantees that the portion of $\mathcal{B}_t$ coming from a neighborhood of x_0 is nonsingular at $\psi(x_0, t)$. However, if the tangent space of $\psi_t(W)$ at $\psi(x_0, t)$ contains the line spanned by U the projection from $\psi_t(W)$ back to M along U will develop singularities (generically $\psi_t(W)$ will foldback as t further increases). However, by 3) this does not happen.

Proof. We consider the Jacobians of both ψ and ψ_t, using the basis $\{\dfrac{\partial}{\partial t}, v_1, \dots, v_n\}$ in the source for $W \times \mathbb{R}$ at (x_0, t) and $\{U_1, v_1, \dots, v_n\}$ for $\mathbb{R}^{n+1}$ at $\psi(x_0, t)$ by translation along U. By 4.2 and 4.3, the transpose Jacobian matrix of ψ has the form

$$\begin{pmatrix} r & 0 \\ t(dr(\mathbf{v}) + rA_{\mathbf{v}}) & (I - tr \cdot S_{\mathbf{v}})^T \end{pmatrix} \tag{4.5}$$

When $t = 0$, as $r(x_0) > 0$, we immediately deduce from (4.5) and the inverse function theorem that ψ is a local diffeomorphism at $(x_0, 0)$.

For 2), we note that the transpose Jacobian matrix of $\psi_t$with respect to the basis $\{v_1, \dots, v_n\}$ for $T_{x_0}M$ and $\{U_1, v_1, \dots, v_n\}$ for $\mathbb{R}^{n+1}$ is

$$\begin{pmatrix} t(dr(\mathbf{v}) + rA_{\mathbf{v}}) & (I - tr \cdot S_{\mathbf{v}})^T \end{pmatrix} \tag{4.6}$$

By the immersion theorem, a sufficient condition that ψ_t is a local embedding is that the matrix (4.6) has rank n, and then $T_{\psi_t(x_0)}\psi(W)$ is spanned by the rows of (4.6). We note, however, that if this matrix (4.6) has rank n but the $n \times n$ matrix $I - tr \cdot S_{\mathbf{v}}$ is singular, then the tangent space of $\psi_t(W)$ at $\psi(x_0)$ will contain the line spanned by U. Thus, to also avoid this, we consider instead the matrix $I - tr \cdot S_{\mathbf{v}}$. Then,

$$I - tr \cdot S_{\mathbf{v}} = -tr \cdot (S_{\mathbf{v}} - \frac{1}{tr} I) \tag{4.7}$$

Hence, (4.7) will be nonsingular for $0 < t \leq 1$ iff $\frac{1}{tr}$ is not an eigenvalue of $S_{\mathbf{v}}$ for $0 < t \leq 1$. Since $\{\frac{1}{tr} : 0 < t \leq 1\} = [\frac{1}{r}, \infty)$, this is equivalent to all positive eigenvalues $\kappa_{r\,i}$ of $S_{\mathbf{v}}$ being less than $\frac{1}{r}$, which is equivalent to (4.4).

Then, not only do we have the local embedding of ψ_t at x_0, but also by (4.5), ψ is a local diffeomorphism at (x_0, t) so the tangent space $T_{\psi_t(x_0)}\psi_t(W)$ is transverse to $U(x_0)$. This implies the nonsingularity of the projection from $\psi_t(W)$ along the lines of U to M at $\psi_t(x_0)$. □

Consider next a point $x_0 \in M_{sing}$ which is not an edge point, and a value $U(x_0)$ pointing into the complementary component C_i. The abstract neighborhood is formed from local manifold components $\{M_j\}$ containing x_0 (M_j have boundaries and corners). For such a non–edge local manifold component M_j with a smooth value U determined by $U(x_0)$, we can define a local radial flow using U and repeat the argument for a local extension of M_j and the smooth value U. We obtain the conclusions of Proposition 4.1 for this extension (see fig. 12).

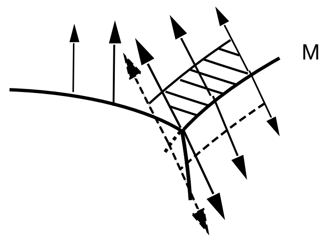

FIGURE 12. Partial Radial Flow near a Singular Point

Of course the extension does not agree with the rest of the Skeletal structure; however, we can then restrict back to M_j to obtain the following.

Corollary 4.3. *Let $x_0 \in M_{sing}$ be a non–edge point. Also, let U be a smooth value on a local (but non–edge) manifold component M_j of x_0. Then, provided the associated radial shape operator for this smooth value satisfies (4.4), then in a neighborhood W of x_0 in M_j, the associated radial flow ψ (and ψ_t) satisfy the three conclusions of Proposition 4.1.*

This analysis thus allows us to partially define the flow at singular points which are not edge points. Next we take the first step in analyzing the radial flow for edge points.

Nonsingularity of the Radial Flow on ∂M. We describe how to extend the preceding to the properties of the radial flow on ∂M (or to local edge manifold components M_j of an edge closure point). We concentrate on an edge point x_0; the argument can be easily adjusted to a local edge manifold M_j of an edge closure point. By assumption, U has a unique value at $x_0 \in \partial M$ which extends smoothly to either a smooth vector field on a neighborhood of x_0 and corresponding to one side of M near x_0. We consider the radial flow map in the following local form $\psi(x,t) = x + t \cdot U(x)$ for x_0; also for fixed t, we let $\psi_t(x) = \psi(x,t)$. As for smooth points, we compute the derivative of the radial flow. For $x_0 \in W$, we choose a basis for edge coordinates near x_0, $v_1, \dots, v_{n-1} \in T_{x_0}\partial M$ a basis for $T_{x_0}\partial M$, and v_n mapping by the edge parametrization to $c \cdot U_1$ with $c \geq 0$. We compute as in (4.1),

$$\frac{\partial \psi}{\partial v_i} = v_i + t\left(\frac{\partial r}{\partial v_i} \cdot U_1 + r \cdot \frac{\partial U_1}{\partial v_i}\right) \tag{4.8}$$

Using (2.4) we obtain

$$\frac{\partial \psi}{\partial v_i} \;=\; (c \cdot \delta_{in} + tra_i + tdr(v_i)) \cdot U_1 - trc_i \cdot \mathbf{n} + \sum_{j=1}^{n-1} (\delta_{ij} - tr \cdot b_{ji}) v_j \qquad (4.9)$$

We can rewrite these equations for $i = 1, \dots, n$ in vector form as

$$\begin{aligned} \frac{\partial \psi}{\partial \mathbf{v}} &= \tilde{A}_U \cdot U_1 - tr \cdot C_U \cdot \mathbf{n} + I_{n-1,1} \cdot \mathbf{v} - tr \cdot B_{U\,\mathbf{v}} \cdot \tilde{\mathbf{v}} \\ &= \tilde{A}_U \cdot U_1 + (I_{n-1,1} - trS^T_{E\,\mathbf{v}}) \cdot \begin{pmatrix} \tilde{\mathbf{v}} \\ \mathbf{n} \end{pmatrix} \end{aligned} \qquad (4.10)$$

where $S_{E\,\mathbf{v}} = (B_{U\,\mathbf{v}}\; C_U)^T$ and as earlier, the column vectors are n–dimensional, with $I_{n-1,1}$, C_U, $B_{U\,\mathbf{v}}$, $\mathbf{v}$, and $\tilde{\mathbf{v}}$ as defined in §2, and lastly $\tilde{A}_U = tr \cdot A_U + c \cdot e_n + tdr(\mathbf{v})$.

Then, with respect to the basis $\{\frac{\partial}{\partial t}, v_1, \dots, v_{n-1}, v_n\}$ in the source for edge coordinates about x_0 and $\{U_1, v_1, \dots, v_{n-1}, \mathbf{n}\}$ in the target, the transpose Jacobian matrix of ψ as a function of (x, t) is

$$\begin{pmatrix} r & 0 \\ \tilde{A}_U & (I_{n-1,1} - tr \cdot S_{E\,\mathbf{v}})^T \end{pmatrix} \qquad (4.11)$$

Then, analogous arguments given for Proposition 4.1 can be repeated to yield the following.

Proposition 4.4. *For a skeletal structure (M, U), let U be a smooth value in a neighborhood W of $x_0 \in \partial M$ (or on a local edge manifold M_j of an edge closure point x_0). Suppose the Edge Conditions are satisfied on this neighborhood. Then, the radial map satisfies the following three properties (see Fig. 13).*

(1) $\psi : W \times (0, 1] \to \mathbb{R}^{n+1}$ *is a local diffeomorphism at* (x_0, t). *Hence, for each* t *with* $0 \leq t \leq 1$
(2) $\psi_t : W \to \mathbb{R}^{n+1}$ *is a local embedding at* x_0
(3) $\psi_t(W)$ *is transverse to the line spanned by* U.

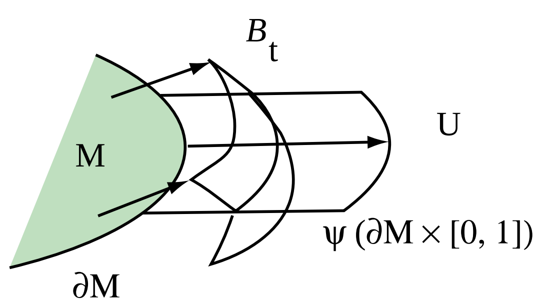

FIGURE 13. The Radial Flow from an edge

Proof. We computed in (4.11) the matrix representation of the transpose Jacobian of ψ with respect to the basis $\{\frac{\partial}{\partial t}, v_1, \dots, v_{n-1}, v_n\}$ for edge coordinates about x_0, and $\{U_1, v_1, \dots, v_{n-1}, \mathbf{n}\}$ in the target. Provided the matrix is nonsingular the inverse function theorem implies ψ is a local diffeomorphism. This then implies (1);

also (2) and (3) follow for $0 < t \leq 1$ as ψ_t is the restriction of the diffeomorphism ψ to $W \times \{t\}$ and the tangent space of $\psi_t(W)$ at $\psi_t(x_0)$ must be transverse to U_1 by the form of the matrix in (4.11). The remaining case $t = 0$ trivially holds.

It remains to see that (4.11) has rank $n+1$; or equivalently that (4.12) has rank n.

$$I_{n-1,1} - tr \cdot S_{E\,\mathbf{v}} \quad = \quad -tr(S_{E\,\mathbf{v}} - \frac{1}{tr} I_{n-1,1}) \tag{4.12}$$

If not then $\frac{1}{tr}$ is a generalized eigenvalue of $(S_{E\,\mathbf{v}}, I_{n-1,1})$. However, just as for the radial curvature condition, the edge condition implies $\frac{1}{tr}$ is not a generalized eigenvalue for $0 < t \leq 1$, a contradiction. □

As a corollary we obtain

Corollary 4.5. *In the situation of Proposition 4.4, for any $0 \leq t \leq 1$,*

(1) $\psi : \partial M \times [0,1] \to \mathbb{R}^{n+1}$ *is a local embedding at (x_0, t);*
(2) $\psi_t : \partial M \to \mathbb{R}^{n+1}$ *is a local embedding at x_0; and*
(3) $\psi_t(\partial M)$ *is transverse within $\psi(\partial M \times [0,1])$ at $\psi_t(x_0)$ to the line spanned by $U(x_0)$ (see Fig. 13).*

Proof. Again the restriction of the diffeomorphism ψ to $\partial M \times [0,1]$ or the diffeomorphism ψ_t to ∂M is again a diffeomorphism. Thus, we can apply Proposition 4.4 to conclude that (1) and (2) hold for $0 < t \leq 1$, while (2) is trivially true for $t = 0$.

For 1) at $t = 0$, we examine the transpose Jacobian matrix for $\psi | \partial M$. This is obtained from (4.11) by removing the bottom row. When $t = 0$, it becomes

$$\begin{pmatrix} r & 0 \\ \tilde{A}'_U & I_{n-1} \end{pmatrix} \tag{4.13}$$

which is nonsingular; hence (1) holds for $t = 0$. Lastly, for 3) we know from Proposition 4.4 that the restriction $\psi | \partial M \times [0,1]$ is an embedding at (x_0, t). The tangent space of the image can be identified with the first n rows of (4.11) (by multiplying them by the column vector with entries $U_1, v_1, \ldots, v_{n-1}$), while that for $\psi_t(\partial M)$ can then be identified with rows 2 through n. It is immediate from the form of (4.11) that the latter is transverse to U_1 within the former. □

Relation between the Grassfire Flow and the Radial Flow. Suppose we are given the Blum medial axis M with radial vector field U of a region with smooth boundary which is generic. Then, M is a Whitney stratified set. We show that (M, U) must satisfy both the radial curvature and edge conditions.

Proposition 4.6. *If (M, U) is the medial axis and radial vector field of a region $\Omega \subset \mathbb{R}^{n+1}$ with generic smooth boundary $\mathcal{B}$, then (M, U) satisfies both the radial curvature and edge conditions.*

Proof. We compute the grassfire flow in terms of the radial representation and show that the condition of the medial axis forces the radial curvature and edge conditions to hold. We consider the the radial curvature condition with the computation for the edge condition being analogous.

Let x_0 be a smooth point of M where a smooth value of U is chosen on a neighborhood W of x_0 (if $x_0 \in M_{sing}$ then we would instead consider a local component M_α of x_0 extended in a neighborhood so x_0 becomes an interior point).

The distance from a point $x' = \psi_t(x_0) = x_0 + tU(x_0)$ to the point on the boundary $x_0' = x_0 + U(x_0)$ corresponding to x_0 is given by

$$x_0 + U(x_0) - (x_0 + tU(x_0)) \quad = \quad (1-t)U(x_0) \quad = \quad (1-t)r(x_0)U_1(x_0)$$

The grassfire flow at time $t' = (1-t)r(x_0)$ will consist of points that are a distance $(1-t)r(x_0)$ from the boundary along U. Thus, we can represent the grassfire flow, but in terms of local coordinates on W by

$$\begin{aligned} \chi(x,t) &= x + U(x) - (1-t)r(x_0)U_1(x) \\ &= x + (r(x) - (1-t)r(x_0))U_1(x) \\ &= x + (r - r_0 + tr_0)U_1(x) \end{aligned} \tag{4.14}$$

where we let $r = r(x)$ and $r_0 = r(x_0)$. From (4.14), we compute the derivative in an analogous manner as for the radial flow.

$$\frac{\partial \chi}{\partial v_i} \;=\; v_i + \frac{\partial r}{\partial v_i} \cdot U_1 + (r - r_0 + tr_0) \cdot \frac{\partial U_1}{\partial v_i}) \tag{4.15}$$

Using (2.2) we can rewrite these equations for $i = 1, \ldots, n$ in vector form as

$$\frac{\partial \chi}{\partial \mathbf{v}} \;=\; (dr(\mathbf{v}) + (r - r_0 + tr_0)\mathbf{1}) \cdot U_1 + (I - (r - r_0 + tr_0) \cdot S_{\mathbf{v}})^T \cdot \mathbf{v} \tag{4.16}$$

where $\mathbf{1}$ is a column vector with all entries equal to 1. Also, $\dfrac{\partial \chi}{\partial t'} = -1$, so the transpose Jacobian matrix of χ with respect to the basis $\{\dfrac{\partial}{\partial t}, v_1, \ldots, v_n\}$ in the source and $\{U_1, v_1, \ldots, v_n\}$ in the target is given by

$$\begin{pmatrix} -1 & 0 \\ (dr(\mathbf{v}) + (r - r_0 + tr_0)\mathbf{1}) & (I - (r - r_0 + tr_0) \cdot S_{\mathbf{v}})^T \end{pmatrix} \tag{4.17}$$

At the point x_0 (4.17) becomes

$$\begin{pmatrix} -1 & 0 \\ (dr(\mathbf{v}) + tr_0\mathbf{1}) & (I - tr_0 \cdot S_{\mathbf{v}})^T \end{pmatrix} \tag{4.18}$$

This matrix is not identical to that for the radial flow ψ. However, it is nonsingular iff $(I - tr_0 \cdot S_{\mathbf{v}})$ is, i.e. iff $\frac{1}{tr_0}$ is not an eigenvalue of $S_{\mathbf{v}}$. For the grassfire flow, χ is nonsingular except at the shock points, i.e. on the medial axis. Hence, for the grassfire flow, $\frac{1}{tr_0}$ is not an eigenvalue of $S_{\mathbf{v}}$ for $0 < t \leq 1$. This is exactly the radial curvature condition. $\square$

5. Existence of a Tubular Neighborhood for a Skeletal Structure

In this section, we prove the existence of a tubular neighborhood for any skeletal structure.

Theorem 5.1. *A skeletal structure (M, U) has a tubular neighborhood.*

Remark 5.2. In fact, the proof for the existence of a tubular neighborhood that we shall give will work if we only assume that M is a skeletal set and U is a radial vector field on M satisfying the local initial conditions (without requiring that the abstract boundaries are homeomorphic to n–disks). However, then in the conclusion, we no longer assert the stronger conclusion that near a point of $\mathcal{B}_t$ coming from a singular point of M, that $\mathcal{B}_t$ need locally be a topological manifold.

To prove the theorem, we adapt the argument from differential topology that a smooth submanifold (without boundary) has a tubular neighborhood, see e.g. [Mu] or [Hi]. We use the local properties of the radial flow for smooth points, non–edge singular points, and edge (closure) points analyzed in §4 to prove that $\tilde{\psi}$ is a local diffeomorphism in a neighborhood of a regular point or a piecewise differentiable local homeomorphism in a neighborhood of either a singular point, or from the "double"of a neighborhood of a point in ∂M. We show these local homeomorphisms imply by a point set topology argument that the global radial flow is a global homeomorphism, yielding the resulting tubular neighborhood of the skeletal structure.

We begin by completing the analysis of the local radial flow in a neighborhood of a point in M_{sing}.

Radial Flow on a Neighborhood of a Point in ∂M. First we consider $x_0 \in \partial M$. By the local edge condition, there is a neighborhood W of x_0 in M such that for a smooth value of U extending to x_0, there is an $\varepsilon > 0$ such that the radial flow $\psi(x,t) : W \times [0,\varepsilon] \to \mathbb{R}^{n+1}$ is one–one. By shrinking W and ε we may assume this holds for each of the two smooth values of U near x_0, and that it continues to hold on the closure of W which we may assume to be compact. Hence, each such local flow is a differentiable homeomorphism which is a diffeomorphism off $W \cap \partial M$. If x_0 is an edge point, an abstract neighborhood $\tilde{W}$ for $(x_0, U(x_0))$ is formed from two copies W_1 and W_2 of W, by identifying the same point of ∂M in the two copies of W (this is the "double"of W, see e.g. [Mu]). Then, the global radial flow $\tilde{\psi} : \tilde{W} \times [0,\varepsilon] \to \mathbb{R}^{n+1}$ is defined using a copy of the radial flow for each smooth value of U near x_0.

Proposition 5.3. *For a possibly smaller neighborhood W and $\varepsilon > 0$, the map $\tilde{\psi}|\tilde{W} \times (0,\varepsilon]$ is a homeomorphism onto its image. Hence, for each t with $0 < t \leq \varepsilon$, the image $\tilde{\psi}(\tilde{W} \times \{t\})$ is a topological manifold which is smooth off the image of ∂M. This is a neighborhood of $\psi_t(x_0)$ in $\mathcal{B}_t$.*

Proof. In fact, we claim that, after possibly further shrinking W, the map $\tilde{\psi}$ is still one–one on $\tilde{W} \times (0,\varepsilon]$. Given this, it follows that $\tilde{\psi}$ restricted to each copy of W is a homeomorphism, and then we deduce from the one–one property that $\tilde{\psi}$ itself is a homeomorphism.

To establish the one-one property, we note from Proposition 4.5 that the radial flow from ∂M in a neighborhood W of a point x_0 for small $\varepsilon > 0$ defines a smooth embedded manifold M' with boundary ∂M such that the tangent spaces of M and M' agree on ∂M. Hence, $W \cap (M \cup M')$ is a piecewise smooth manifold which is weakly C^1 on ∂M. Then, we use a lemma which is stronger than is needed here but which will be of use again shortly in this stronger form.

Lemma 5.4. *Let M_1 and M_2 be smooth n–manifolds with boundaries in $\mathbb{R}^N$. Suppose there is a neighborhood W of $x_0 \in \partial M_1, \partial M_2$, such that $W \cap \partial M_1 = W \cap \partial M_2$, and for $x \in W \cap \partial M_i$, $T_x M_1 = T_x M_2$. Then, $W \cap (M_1 \cup M_2)$ is C^1 on $W \cap \partial M_i$.*

Before proving this Lemma, we first use it to complete the argument regarding $\tilde{\psi}$. By the Lemma, $L = W \cap (M \cup M')$ is a C^1 submanifold. Hence, in a neighborhood V of x_0 in $\mathbb{R}^{n+1}$, $V \backslash L$ consists of two connected components. We may shrink W to W_1 and ε to ε_1 such that $\psi(W_1 \times [0,\varepsilon_1]) \subset V$. Then, for fixed $x_1 \in W_1$, as $\psi(x_1,t)$

begins for small t on one side of L and ψ is one–one on $W \times [0, \varepsilon]$, $\psi(x_1, t)$ does not intersect L for $0 < t \leq \varepsilon_1$. Thus, $\psi(x_1, t)$ remains on one side of L for $0 < t \leq \varepsilon_1$. As this is true for all $x_1 \in W_1$, we conclude the image $\psi(W_1 \times [0, \varepsilon_1])$ is one side of L. Thus, on a smaller neighborhood W_1 and smaller $\varepsilon_1 > 0$, the two radial flows for each smooth value map to opposite sides of L, Hence, the map $\tilde{\psi}$ is one–one on $\tilde{W} \times (0, \varepsilon]$ for appropriately smaller W and ε. □

Proof of Lemma 5.4. Let H^n denote the half space $\{\mathbf{x} \in R^n : x_n \geq 0\}$. There are coordinate charts $\chi_i : V_i \to H^n$ on neighborhoods V_i of x_0 in each M_i, where we assume $V_i \cap \partial M_i$ agree. By assumption, the vector field $-\dfrac{\partial}{\partial x_n}$ on H^n corresponds to a smooth vector field on V_1, which on ∂M_1 is tangential to M_2 and points into M_2 transverse to ∂M_2. This corresponds via χ_2 to a vector field on $\mathbb{R}^{n-1}$ of the form $\xi = \sum_{i=1}^{n} a_i(x_1, \dots, x_{n-1}) \dfrac{\partial}{\partial x_i}$ for with $a_i > 0$ on $\chi_2(V_2) = V_2'$. Then, we can extend ξ to a neighborhood in H^n by translation in the x_n direction. Now, we may choose flow box coordinates x' corresponding to ξ for H^n, so that $(x_1', \dots, x_{n-1}')$ are local coordinates for $\mathbb{R}^{n-1}$. We shrink V_2 so we may view these coordinates as a local diffeomorphism $\varphi_1 : V_2' \to H^n$. Let $\varphi_2 = (\chi_2^{-1} \circ \chi_1 | \mathbb{R}^{n-1}) \times id_{\mathbb{R}}$. Then, we define a local coordinate chart $\chi' : V_1 \cup V_2 \to \mathbb{R}^n$ by $\chi' | V_1 = \chi_1$ and $\chi' | V_2 = \varphi_1 \circ \chi_2 \circ \varphi_2$. They agree on $\partial M_i \cap V_i$. Also, χ' is differentiable with respect to the original coordinates in the $\mathbb{R}^{n-1}$ direction, and the normal derivatives are continuously differentiable, so it is C^1. □

Radial Flow on a Neighborhood of a Singular Point. Next, we consider the radial flow in a neighborhood of a point (x_0, U_0) of $\tilde{M}$ corresponding to a singular point $x_0 \in M_{sing}$ which is not an edge point. This means we also possibly allow x_0 to be an edge closure point. Suppose U_0 points into a complementary component C_i, then by property 2) for the radial vector field, there is a neighborhood W of x_0 and an $\varepsilon > 0$ so that the map $(W \cap \partial C_i) \times [0, \varepsilon] \to \mathbb{R}^{n+1}$, sending $(x, t) \mapsto x + tU(x)$ maps into C_i.

Locally in the neighborhood W of x_0, we let $\{M_{\alpha_1}, \dots, M_{\alpha_m}\}$ denote the finite set of local manifold components for x_0 having smooth values pointing into C_i. If necessary, we subdivide each M_{α_i} so any such has at most a single boundary edge which is also an edge of ∂M. Also, by the Whitney conditions, M is stratified trivial along the stratum M_γ of x_0, hence, we may assume that in a neighborhood of x_0, each M_{α_i} contains M_γ as a boundary facet.

For any M_{α_i} with a boundary edge which is an edge of M, both smooth values on M_{α_i} will point into C_i. Then, in an abstract neighborhood $\tilde{W}$ of x_0 each M_{α_i} will have one copy unless its boundary contains an edge of M extending into the interior of the closure of C_i. We denote the smooth manifold components in $\tilde{W}$ by M_{β_i}.

Then, we establish an analogue of Proposition 5.3.

Proposition 5.5. *For a possibly smaller neighborhood W and $\varepsilon > 0$, the map $\tilde{\psi} | \tilde{W} \times (0, \varepsilon]$ is a homeomorphism onto its image. Moreover, for each t with $0 < t \leq \varepsilon$, the image $\tilde{\psi}(\tilde{W} \times \{t\})$ is a topological manifold which is smooth off the image of M_{sing}. This is a neighborhood of $\psi_t(x_0)$ in $\mathcal{B}_t$.*

Proof. First we establish the homeomorphism property and then deduce that the image for fixed t is a topological manifold in a neighborhood of the image of x_0.

First, by the local diffeomorphism properties from Propositions 4.1, 4.5, 5.3, and Corollary 4.3, we know that for sufficently small $\varepsilon > 0$, the restriction of $\tilde{\psi}$ to any M_{β_i} is a diffeomorphism. Hence, we can choose a single small $\varepsilon > 0$ so that this is simultaneously true for all M_{β_i}.

We claim that we may shrink W and ε so that these piecewise homeomorphisms fit together to give a homeomorphism in a neighborhood of (x_0, U_0). First, we show $\tilde{\psi}$ is locally one–one. It is enough for this to show that for each pair M_{β_i} and M_{β_j}, that there is an $\varepsilon > 0$ such that $\tilde{\psi}|(N_\varepsilon|(M_{\beta_i} \cup M_{\beta_j}))$ is one–one off $\tilde{W}$. Then we choose a minimum ε for all pairs.

If M_{β_i} and M_{β_j} are both copies of the same M_{α_i}, then they meet along an edge of M. Then, by Proposition 5.3, there is an $\varepsilon > 0$ so the restriction to $N_\varepsilon|(M_{\beta_i} \cup M_{\beta_j})$ is one–one off $\tilde{W}$. Second, suppose instead that M_{β_i} and M_{β_j} correspond to different M_{α_i} and M_{α_j} in W. Then, $M_{\alpha_i} \cap M_{\alpha_j}$ is a facet of each which contains the stratum of x_0. By the local separation assumption for skeletal structures, for $\varepsilon > 0$, the flow on $\cup_i \partial M_{\alpha_i}$ defines an n–dimensional embedded Whitney stratified set L which separates the M_{α_i} into distinct components. Hence, we can choose a neighborhood of x_0 in M_{α_i} such that its intersection with the interior of M_{α_i}, W_i is path–connected. Then, $W_i \times [0, \varepsilon]$ is path–connected, and intersects the component of M_{α_i}. To also intersect the component of M_{α_j} it would have to pass through the L. In particular, for a small enough W_i by continuity it would have to pass through the image of ∂M_{α_i}. However, the flow is one–one on M_{α_i}, a contradiction.

Thus, for small enough W and $\varepsilon > 0$, the radial flow is one–one on $(N_\varepsilon|\tilde{W})\backslash\tilde{W}$. Then, the restriction of the flow to each $M_{\beta_i} \times (0, \varepsilon]$ is a diffeomorphism, and each M_{β_i} is closed in $\tilde{W}$. Then, the $M_{\beta_i} \times (0, \varepsilon]$ give a finite decomposition of $(N_\varepsilon|\tilde{W})\backslash\tilde{W}$ by closed subsets and the restriction $\tilde{\psi} : M_{\beta_i} \times (0, \varepsilon] \to \tilde{\psi}(N_\varepsilon|\tilde{W})\backslash\tilde{\psi}(\tilde{W})$ is closed. It follows that $\tilde{\psi}$ maps $(N_\varepsilon|\tilde{W})\backslash\tilde{W}$ homeomorphically onto its image.

Then, for each $0 < t \leq \varepsilon$, the restriction of $\tilde{\psi}$ to each $S_t = \{(x, tU(x)) \in N_\varepsilon : x \in \tilde{W}\}$ is also a homeomorphism onto its image. Also, S_t is homeomorphic to the abstract boundary $\partial\tilde{C}_i$, which by assumption is homeomorphic to an n–disk. However, this is exactly the image of $\tilde{\psi}|\tilde{W}$ in $\mathcal{B}_t$, proving that its image in $\mathcal{B}_t$ is locally a topological manifold near $\psi_t(x_0)$.

Lastly, we assert that the image of $N_\varepsilon|\tilde{W}$ is a neighborhood of x_0 in $\bar{C}_i$. We choose a small ball $B_\delta(x_0)$ and intersect it with each component of the complement of L. As M_{α_i} is a manifold with boundary and U extends smoothly up to the boundary, both extend smoothly in a small neighborhood of the boundary. Hence, the flow defines a diffeomorphism in a neighborhood of x_0, so for sufficiently small $\delta > 0$ the neighborhood contains the intersection of $B_\delta(x_0)$ with the component corresponding to M_{α_i} in $\bar{C}_i$. Taking the mininmum of δ over all M_{α_i} in $\partial\tilde{C}_i$, and then over all C_i for x_0, we conclude that $B_\delta(x_0)$ is in the union of $\tilde{\psi}(N_\varepsilon|\tilde{W})$, completing the proof of the proposition. $\square$

Construction of a Tubular Neighborhood of a Skeletal Structure. We are now prepared to prove the existence of the tubular neighborhood. Given the local results we have already established, we reduce to the following Lemma from point set topology.

Lemma 5.6. *Suppose that $f : X \times Z \to Y$ is a continuous map, with X, Y, and Z metric spaces, and X compact and Z locally compact. Let $z_0 \in Z$. Suppose the restriction $f : X \times \{z_0\} \to Y$ is a finite to one map. Also, suppose that for each $y \in Y$, if $f^{-1}(y) \cap (X \times \{z_0\}) = \{(x_1, z_0) \dots, (x_k, z_0)\}$, then there are open neighborhoods W_i of x_i in X and an $\varepsilon > 0$ (that depends on y) such that $f : (\cup_i W_i) \times (B_\varepsilon(z_0) \backslash \{z_0\}) \to Y$ is a homeomorphism onto its image and is in the complement of $f(X \times \{z_0\})$. Then there exists an $\varepsilon > 0$ such that $f : X \times (B_\varepsilon(z_0) \backslash \{z_0\}) \to Y$ is a homeomorphism onto its image which is in the complement of $f(X \times \{z_0\})$.*

Before proving the Lemma, we complete the proof of the existence of the tubular neighborhood.

Proof of Theorem 5.1. We apply the Lemma with $X = \tilde{M}$, $Z = [0, \varepsilon]$ with $z_0 = 0$, and $Y = \mathbb{R}^{n+1}$, to conclude that $\tilde{\psi}|(N_\varepsilon \backslash \tilde{M})$ is a homeomorphism. Then the restriction $\tilde{\psi}|(\tilde{M} \times \{t\})$ is a homeomorphism onto $\mathcal{B}_t$. Then, we can apply Proposition 5.3 and Lemma 5.4 as no other points outside of a neighborhood $\tilde{W}$ map to a neighborhood of $\psi_t(x_0)$, it follows that $\mathcal{B}_t$ is a topological manifold in a neighborhood of the images of points from M_{sing} as well as from points in M_{reg}.

Finally, we have shown that $|N| = \tilde{\psi}(N_\varepsilon)$ contains a neighborhood of any singular point as well as any smooth point of M. Hence, $|N|$ is a topological neighborhood of M, completing the proof □

Proof of Lemma 5.6. We first claim there is an $\varepsilon > 0$ such that the map $f : X \times (B_\varepsilon(z_0) \backslash \{z_0\}) \to Y$ is one–one. If not then there are sequences of points $x_i, x'_i \in X$, $z_i, z'_i \in B_{\frac{1}{i}}(z_0) \backslash \{z_0\}$ such that $f(x_i, z_i) = f(x'_i, z'_i)$, but $(x_i, z_i) \neq (x'_i, z'_i)$ for all i. Then, by compactness of X and restricting to subsequences, we may assume $\{x_i\}$ converges to x_0 and $\{x'_i\}$ converges to x'_0. Since $\{z_i\}$ converges to z_0, by the continuity of f, $f(x_0, z_0) = f(x'_0, z_0) = y_0$, say. By assumption, there are neighborhoods W_1 of x_0 and W_2 of $x'_0$and an $\varepsilon > 0$ such that the map $(W_1 \cup W_2) \times (B_\varepsilon(z_0) \backslash \{z_0\}) \to Y$ is a homeomorphism. This is a contradiction as for sufficiently large i, $x_i \in W_1$, $x'_i \in W_2$, $z_i \in B_\varepsilon(z_0)$, but $f(x_i, z_i) = f(x'_i, z'_i)$. Thus, there is an $\varepsilon > 0$ such that $f : X \times (B_\varepsilon(z_0) \backslash \{z_0\}) \to Y$ is one–one.

Furthermore, by a similar argument we can show that $f : X \times (B_\varepsilon(z_0) \backslash \{z_0\}) \subset Y \backslash f(X \times \{z_0\})$.

Finally to see $f : X \times (B_\varepsilon(z_0) \backslash \{z_0\})$ is a homeomorphism onto its image, we show it is a closed map. Choose $0 < \varepsilon_1 < \varepsilon$. Then, $f|X \times \bar{B}_{\varepsilon_1}(z_0)$ is a closed map. Also, by f being one–one on $X \times \bar{B}_\varepsilon(z_0)$ with image disjoint from $f(X \times \{z_0\}$

$$f(X \times (\{z_0\} \cup (\bar{B}_{\varepsilon_1}(z_0) \backslash B_{\varepsilon_1}(z_0)))) \cap f(X \times B_{\varepsilon_1}(z_0) \backslash \{z_0\}) = \emptyset$$

Hence, for any (relatively) closed subset $A \subset X \times B_{\varepsilon_1}(z_0) \backslash \{z_0\}$, $f(A) = f(\bar{A}) \cap f(X \times B_{\varepsilon_1}(z_0) \backslash \{z_0\})$. Thus, $f|X \times B_{\varepsilon_1}(z_0) \backslash \{z_0\}$ is a closed map, so we replace ε by ε_1. □

Lastly, we use results from the proof of the preceding to prove Lemma 1.11.

Proof of Lemma 1.11. We recall that C_i is a complementary component for x_0 with C'_i the associated complementary component to the link of M at x_0 in a small sphere about x_0. We let $\varphi : C'_i \times (0, \varepsilon] \to C_i$ denote the diffeomorphism which is the restriction of the stratified homeomorphism of the cone $c(S_\varepsilon(x_0), L) \simeq (B_\varepsilon(x_0), M \cap$

$B_\varepsilon(x_0)$) following from the conical structure of the Whitney stratified set M about x_0. Thus, C_i is contractible iff C_i' is.

Then, from the proof of the existence of tubular neighborhoods, there is a $\delta > 0$ such that $B_\delta(x_0) \subset |N|$ (by Remark 5.2, this applies to our given (M, U)). Then, there is an $\varepsilon' > 0$ used for the tubular neighborhood, so that the radial flow map $\tilde{\psi} : \partial\tilde{C}_i \times (0, \varepsilon'] \to C_i$ maps homeomorphically to $|N| \cap C_i$. Now, by assumption, $\partial\tilde{C}_i$ is homeomorphic to a cone with vertex x_0. Hence, $\partial\tilde{C}_i \times (0, \varepsilon']$ is contractible.

Then, we first use the cone structure to give a deformation retract of C_i to $C_i \cap B_\delta(x_0)$. Then, we use the contraction of $|N| \cap C_i$ restricted to $C_i \cap B_\delta(x_0)$ to construct a contraction of $C_i \cap B_\delta(x_0)$ to a point within $|N| \cap C_i \subset C_i$. This gives the required contraction of C_i to a point. □

6. Smoothness of the Boundary of the Skeletal Structure

We are ready to prove the smoothness of the boundary using the global radial flow from the tubular neighborhood. It follows that the radial flow is a well defined mapping from any level set $\mathcal{B}_t$ of $|N|$. The second step in the proof of smoothness of the boundary is to apply the results on the radial flow as a map $\mathcal{B}_\varepsilon \to \mathcal{B}_t$ for any t with $\varepsilon < t \leq 1$. Then, under the assumptions that the three conditions for smoothness hold, we will show that the radial flow is: i) smooth and a local diffeomorphism on points which do not come from the image of M_{sing}; ii) a local piecewise differentiable homeomorphism on a neighborhood at points in M_{sing}; and iii) at $t = 1$, $\mathcal{B}_1 = \mathcal{B}$ is weakly C^1 on points which are images of M_{sing}. To establish i), we use the radial curvature condition. To establish ii), we also use the radial curvature conditions for points in M_{sing} but not in ∂M; while we use the Edge Conditions for points in ∂M. Lastly, for iii), we use the Compatibility Condition.

We first establish a consequence of the compatibility condition.

Lemma 6.1. *Let (M, U) be a skeletal structure. Suppose that M_α is a local manifold component of x_0 on which is defined a smooth value of U. Suppose that either $\frac{1}{r}$ is not an eigenvalue of S_{rad} if M_α is a nonedge component or $\frac{1}{r}$ is not a generalized eigenvalue of $(S_{E\,\mathbf{v}}, I_{n-1,1})$ if M_α is an edge component. If the associated compatibility 1–from η vanishes at x_0, then $U(x_0)$ is orthogonal to the portion of the boundary $\mathcal{B}$ (given by $\psi_1(M_\alpha)$) at $\psi_1(x_0)$.*

Proof. By the proofs of Proposition 4.1 or 4.4, as $\frac{1}{r}$ is not an eigenvalue of S_{rad} or generalized eigenvalue of $(S_E, I_{n-1,1})$, then ψ_1 is a local diffeomorphism on M_α in a neighborhood W of x_0. For $v \in T_{x_0}M$, we compute the dot product $\dfrac{\partial \psi_1}{\partial v} \cdot U_1$ using the first line of (4.1).

$$\frac{\partial \psi_1}{\partial v} \cdot U_1 = v \cdot U_1 + dr(v)(U_1 \cdot U_1) + r(\frac{\partial U_1}{\partial v} \cdot U_1)$$

since differentiating $U_1 \cdot U_1 = 1$ implies $\dfrac{\partial U_1}{\partial v} \cdot U_1 = 0$, we obtain

$$\frac{\partial \psi_1}{\partial v} \cdot U_1 = v \cdot U_1 + dr(v) = \eta_U(v) \tag{6.1}$$

Thus, $\dfrac{\partial \psi_1}{\partial v}$ being orthogonal to $U_1(x_0)$ is equivalent to $\eta_{U\,(x_0)}(v) = 0$.

Hence, the translate of $U_1(x_0)$ along the line spanned by $U(x_0)$ is orthogonal to $T_{x'_0}\mathcal{B}$ where $x'_0 = \psi_1(x_0)$. $\square$

Let (M, U) be a skeletal structure, with a tubular neighborhood $|N|$ which is the image of N_ε. By the properties of tubular neighborhoods, given $x_0 \in M$ with value U_0, there is an ε_0 with $0 < \varepsilon_0 < \varepsilon$ and a $\delta > 0$ such that if $x'_0 = \tilde{\psi}(x_0, \varepsilon_0 \cdot U_0)$ then $B_\delta(x'_0) \subset |N| \backslash M$. Then, the radial flow can be defined on $B_\delta(x'_0)$ because each point in $B_\delta(x'_0)$ has a unique value U associated to it via the radial flow on N_ε. We still write this radial flow by $\psi(x'_1, t) = x_1 + (t + t_1)U$ where $x'_1 = \tilde{\psi}(x_1, t_1 \cdot U)$. Also, as earlier, we let $\psi_t(x) = \psi(x, t)$.

To prove theorem 2.5 we will use the following Proposition.

Proposition 6.2. *Suppose (M, U) is a skeletal structure which satisfies the three conditions: radial curvature condition, edge condition, and compatibility condition. Let $x_0 \in M$ be as in the preceding situation so $x'_0 = \tilde{\psi}(x_0, \varepsilon_0 \cdot U_0)$. Then, for each t, $0 < t \leq 1 - \varepsilon_0$, the map $\psi_t : B_\delta(x'_0) \to \mathbb{R}^{n+1}$ is a local homeomorphism; and hence for fixed t, the restriction $\psi_t | \mathcal{B}_{\varepsilon_0} \cap B_\delta(x'_0)$ is a homeomorphism onto its image.*

We first see how the Proposition implies the Theorem.

Proof of Theorem 2.5. We first consider the restriction of the radial flow from $\psi_{1-\varepsilon_0} : \mathcal{B}_{\varepsilon_0} \to \mathcal{B}$. By Proposition 6.2, the restriction of $\psi_{1-\varepsilon_0}$ to a sufficently small neighborhood of a point $x'_0 \in \mathcal{B}_{\varepsilon_0}$ is a homeomorphism to its image in $\mathcal{B}$. In the case x'_0 comes from a point in M_{reg}, then by the radial curvature condition and Proposition 4.1, $\psi_{1-\varepsilon_0}$ is a local diffeomorphism.

Next, suppose x'_0 comes from a point in M_{sing}. We claim the compatibility condition ensures that the limiting tangent planes from different directions will agree, establishing weak C^1 regularity of $\mathcal{B}$ at such points. We show that the limiting tangent planes are orthogonal to U_0. For a smooth value of U at a point x' of M_{reg}, and $v \in T_{x'}M$, we computed the dot product(6.1) in the proof of Lemma 6.1

$$\frac{\partial \psi_1}{\partial v} \cdot U_1 = v \cdot U_1 + dr(v) = \eta_{U\,(x_0)}(v) \tag{6.2}$$

Hence, when $t = 1$, as $x' \to x_0$, if $v' \to v$, then by the continuity of η_U on a local component of x_0, $\eta_{U\,(x_0)}(v) = \lim \eta_{U\,(x')}(v')$. Thus, the limiting tangent plane $\lim d\psi_1(T_{x'}M)$ will be orthogonal to U_0 if $\eta_{U\,(x_0)}(v) = 0$ for all v in the limiting tangent plane. This will be true for all limiting tangent planes from all neighboring components of the stratum of x_0. Hence, we conclude the limiting tangent planes from all neighboring components agree; so that ψ_1 is weakly C^1 at each such $x_1 = \psi_1(x_0)$. By Lemma 5.4, $\mathcal{B}$ is C^1 on the strata of codimension 1.

Next, we remark that the radial flow $\mathcal{B}_{\varepsilon_0} \to \mathcal{B}$ is globally one–one iff the radial flow $\tilde{\psi} : \tilde{M} \times \{1\} \to \mathcal{B}$, sending $(x_0, U_0) \mapsto x_0 + U_0$ is one–one. In this case these are global homeomorphisms as both $\tilde{M} \simeq \mathcal{B}_{\varepsilon_0}$ are compact.

Finally we consider the global failure of $\psi_{1-\varepsilon_0} : \mathcal{B}_{\varepsilon_0} \to \mathcal{B}$ (or equivalently $\psi_1 : \tilde{M} \to \mathcal{B}$) being one–one. We know $\psi_t : \tilde{M} \to \mathcal{B}_t$ is one–one for $0 < t \leq \varepsilon$. Hence, if $\Delta\mathbb{R}^{n+1} \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ is the diagonal, then $\tilde{\psi} \times \tilde{\psi} : \tilde{M} \times \tilde{M} \times [0, 1] \to \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ sending $(x, x', t) \mapsto (\tilde{\psi}(x, t), \tilde{\psi}(x', t))$ satisfies

$$(\tilde{\psi} \times \tilde{\psi})^{-1}(\Delta\mathbb{R}^{n+1}) \cap (\tilde{M} \times \tilde{M} \times (0, \varepsilon]) \quad \subset \quad \Delta\tilde{M} \times (0, \varepsilon]$$

Then, $(\tilde{\psi}\times\tilde{\psi})^{-1}(\Delta\mathbb{R}^{n+1})$ is closed. Also, by Proposition 6.2 there is a neighborhood $\mathcal{V}$ of $\Delta\tilde{M}\times(0,1]$ such that $(\tilde{\psi}\times\tilde{\psi})^{-1}(\Delta\mathbb{R}^{n+1})\cap\mathcal{V}=\Delta\tilde{M}\times(0,1]$. Thus, if ψ_1 is not one–one, then there is a smallest $t>\varepsilon$ such that ψ_t is not one–one, and moreover, there must be $(x,x')\in\tilde{M}\times\tilde{M}$ which is outside a neighborhood of $\Delta\tilde{M}$ such that $\psi_t(x)=\psi_t(x')$. This corresponds to the radial flow of x' first intersecting the radial flow in a neighborhood of x for the first time at time t, but not meeting in another neigborhood before then. This completes the proof. □

Proof of Proposition 6.2. We first make several preliminary remarks. Let (M,U) satisfy the three conditions: radial curvature condition, edge condition, and compatibility condition. Suppose $x_0\in M_\alpha$, a local manifold component. If M_α is not an edge manifold component, then the radial curvature condition is satisfied at x_0 for any smooth value. By the continuity of both r and S_{rad} for a the smooth value of U, there is a neighborhood W of x_0 and a $r_1>r$ so that the radial curvature condition still holds with r_1 in place of r at all points of W. Then, by the continuity of the radial flow, there is a $\tau>0$ such that the radial flow on W still satisfies the conclusion of Proposition 4.1 for $t<1+\tau$. Similarly, if M_α is a edge manifold component, then the edge condition is satisfied at x_0 for the smooth value which extends to the edge. There is again a neighborhood W of x_0 and a $r_1>r$ so that the edge condition still holds with r_1 in place of r at all points of $W\cap\partial M$. Likewise, there is a $\tau>0$ such that the radial flow on W still satisfies the conclusion of Proposition 4.4 for $t<1+\tau$.

In either case, for x_0, we can choose a single $\tau>0$ which satisfies the preceding for all local manifold components for x_0. Then, we can choose $\varepsilon_1<\varepsilon_0<\varepsilon$ and then a $\delta>0$ small enough so that also $B_\delta(x_0')\subset\tilde{\psi}((N_\varepsilon\backslash\bar{N}_{\varepsilon_1})|\tilde{W})$ and if $x_1'=\tilde{\psi}(x_1,t\cdot U_0)\in B_\delta(x_0')$, then $t<\varepsilon_0+\tau$.

As the radial flow $\psi_t:B_\delta(x_0')\to\mathbb{R}^{n+1}$ is continuous and $B_\delta(x_0')$ is open, the proof will follow from invariance of domain by showing that $\psi_t|B_\delta(x_0')$ is one–one for $0\le t\le 1-\varepsilon$. We will establish this by induction on the codimension of the stratum of x_0. We first show that ψ_t is one–one on the image of each local manifold component in $B_\delta(x_0')$ and then show that when these individual pieces are put together it remains one–one.

We will make use several times of the following consequence of Lemma 5.6. We assume that the skeletal structure (M,U) satifies the three conditions. We consider $x_0\in M$ belonging to the stratum M_γ with value U_0 which extends smoothly to M_γ and a local manifold component M_α. We let τ and W be given by the preceding discussion.

Lemma 6.3. *After possibly shrinking W, $\psi:(W\cap M_\alpha)\times[\varepsilon_1,1+\tau]\to\mathbb{R}^{n+1}$ is one–one. An analogous result holds for M_γ in place of M_α.*

Proof. The map $\psi:\{x_0\}\times[\varepsilon_1,1+\tau]\to\mathbb{R}^{n+1}$ is the parametrization of the straight line $\ell=\{x_0+t\cdot U:t\in[\varepsilon_1,1+\tau]\}$ and hence is one–one. Furthermore by either Proposition 4.1 or 4.4 ψ is a local diffeomorphism at each point of $\{x_0\}\times[\varepsilon_1,1+\tau]$. Then, we can apply Lemma 5.6 (with X and Z denoting $[\varepsilon_1,1+\tau]$ and $W\cap M_\alpha$ respectively) to $\psi|(W\cap M_\alpha)\times[\varepsilon_1,1+\tau]$ to conclude there is a neighborhood W' of x_0 in W such that $\psi:(W'\cap M_\alpha)\backslash\{x_0\}\times[\varepsilon_1,1+\tau]\to\mathbb{R}^{n+1}$ is one–one with image disjoint from ℓ. Thus, after adding $x_0\times[\varepsilon_1,1+\tau]$, ψ will remain one–one. □

We return to the proof of Proposition 6.2.

If x_0 is a smooth point of M, then we can choose $x_0 \in \mathrm{int}(M_\alpha)$. Hence, in Lemma 6.3 W is a neighborhood of x_0, implying the result in this case.

Next suppose x_0 belongs to a stratum of codimension one. First suppose $x_0 \in \partial M$. We apply Lemma 6.3 to the stratum ∂M to conclude there is a neighborhood W' of x_0 in ∂M such that $\psi : W' \times [\varepsilon_1, 1+\tau] \to \mathbb{R}^{n+1}$ is a global diffeomorphism. Let L denote the image, which is an embedded n–manifold (with boundary), with line $\ell = \psi(\{x_0\} \times [\varepsilon_1, 1+\tau])$ a submanifold (recall Fig. 13). Second, we apply Lemma 6.3 to x_0 viewed as a point in M for smooth values of U corresponding to each side of M. We obtain one–one maps $\psi_i : W_i \times [\varepsilon_1, 1+\tau] \to \mathbb{R}^{n+1}$, for $i = 1, 2$, corresponding to the smooth values for each side. Let $K_i = \psi_i(W_i \times [\varepsilon_1, 1+\tau])$. Then, on any point $x \in \ell$ which is not an endpoint, there is a neighborhood of x in each K_i homeomorphic to a closed half space with boundary in L. Thus, the union of these neighborhoods is a neighborhood of x in $\mathbb{R}^{n+1}$. Hence, $K = K_1 \cup K_2$ is a neighborhood in $\mathbb{R}^{n+1}$ for the sub-line segment $\ell' = \{x_0 + t \cdot U : t \in [\varepsilon_0, 1]\}$. We can choose a cylindrical neighborhhood C of ℓ' in K such that $C \backslash L$ has two connected components. Choose a small $\delta > 0$ so that $\psi_t(B_\delta(x_0')) \subset C$ for $0 \leq t \leq 1 - \varepsilon_0$. If $\psi_t | B_\delta(x_0')$ is not one–one, then there must be $x_i \in K_i \cap B_\delta(x_0')$ for which $\psi_t(x_1) = \psi_t(x_2)$. The $\psi_t(x_i)$ are in different components of $C \backslash L$, and cannot cross L (as ψ_t is one–one on each K_i) so they must remain on opposite sides of L, a contradiction.

The other codimension one strata consist of non-edge closure singular points $x_0 \in M_\gamma$. Let U_0 be a value at x_0 which points into a local complementary component C_i. As M_γ has codimension one, the abstract boundary consists of two local manifold components M_1 and M_2, with intersection the stratum M_γ of x_0 in a sufficiently small neighborhood W of x_0.(see Fig. 14).

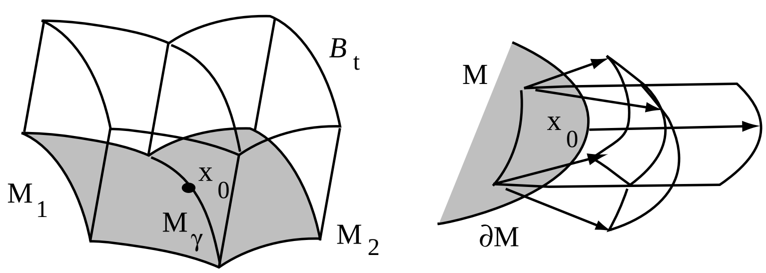

FIGURE 14. Radial Flow on Strata of Codimension One

Now, we repeat the argument used for edge points by applying Lemma 6.3 to construct L from the flow on M_γ, and K_i from the flows on M_i. On each piece, the flow is one–one by Lemma 6.3. We can repeat the argument with the cylindrical neighborhood C of ℓ'. By the local separation property (i.e. local initial condition 1) of Definition 1.7), the components of $K_i \cap C$ are initially on opposite sides of L and hence must remain so.

Lastly, we assume that the result holds for x_0 in strata of codimension $< k$, and let x_0 be in a statum M_γ of codimension k and U_0 a value at x_0. U_0 points into a local complementary component C_i. Let $M_{\beta_i}, i = 1, \ldots, m$, be the local manifold components belonging to the abstract boundary of C_i. We may apply Lemma 6.3 to each individual local manifold component and to the stratum M_γ. To obtain K_i for each M_i and L from M_γ, with the radial flow one–one for each M_i and for L. The K_i and L are manifolds with boundaries and corners. Suppose ψ_t is not one–one on the union for some t with $0 \leq t \leq 1 - \varepsilon_0$, then there are x_i in two distinct K_i, which we call K_1 and K_2 for which $\psi_t(x_1) = \psi_t(x_2)$. If x_1 were a boundary point of K_1, then by the induction assumption, ψ_t is a local homeomorphism at each x_i so there are other points, not boundary points of K_i which map to the same point. Hence, we may assume x_1 is an interior point of $K_1 \cap B_\delta(x'_0)$. Then, by the local separation property again, the boundary of K_i separates it from the images of the other M_i. Hence, $\psi_t(x_2)$ begins outside this boundary when $t = 0$, and hence must remain outside it, a contradiction.

□

Thus, the proof of Proposition 6.2 and hence Theorem 2.5 are complete.

Remark 6.4. We also can see from the proof that for those values t for which ψ_t remains one–one, the level set $\mathcal{B}_t$ is a Whitney stratified set. We have seen that the strata of M are mapped diffeomorphically to manifolds in $\mathcal{B}_t$. As closures are preserved, the axiom of the Frontier is satisfied. It is enough to see Whitney's condition b is satisfied. Let $x_0 \in M_\gamma$, and let $x_i \in M_\beta$ be a sequence converging to x_0. A neighborhood of x_0 consists of a finite number of local manifold components M_{α_i} whose boundaries contain the positive codimension strata near x_0. Thus, we can find a subsequence x_{i_j} contained in a single component M_{α_ℓ}. Then, both M_{α_ℓ} and the smooth value U on M_{α_ℓ} extend to an open manifold W' containing x_0 and a smooth vector field U' on it. By Lemma 6.3, on a smaller neighborhood of x_0, ψ_t is a diffeomorphism. Hence, the pair consisting of the component of the stratum M_β in M_{α_ℓ} and M_γ satisfies Whitney condition b). Thus, under ψ_t so do their images. Hence, the condition b) is also satisfied for the image strata on $\mathcal{B}_t$. Hence, $\mathcal{B}_t$ is also a Whitney stratified set.

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