DETECTING THE CHARACTERISTIC COHOMOLOGY OF MATRIX SINGULARITIES

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Abstract. For a germ of a variety $\mathcal{V}, 0 \subset \mathbb{C}^N, 0$, a singularity $\mathcal{V}_0$ of "type $\mathcal{V}$" is given by a germ $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^N, 0$ which is transverse to $\mathcal{V}$ in an appropriate sense so that $\mathcal{V}_0 = f_0^{-1}(\mathcal{V})$. If $\mathcal{V}$ is a hypersurface germ, then so is $\mathcal{V}_0$, and by transversality $\mathrm{codim}_{\mathbb{C}}\mathrm{sing}(\mathcal{V}_0) = \mathrm{codim}_{\mathbb{C}}\mathrm{sing}(\mathcal{V})$ provided $n > \mathrm{codim}_{\mathbb{C}}\mathrm{sing}(\mathcal{V})$. So $\mathcal{V}_0, 0$ will exhibit singularities of $\mathcal{V}$ up to codimension n.

For singularities $\mathcal{V}_0, 0$ of type $\mathcal{V}$, we introduce a method to capture the contribution of the topology of $\mathcal{V}$ to that of $\mathcal{V}_0$. It is via the "characteristic cohomology" of the Milnor fiber (for $\mathcal{V}, 0$ a hypersurface), and complement and link of $\mathcal{V}_0$ (in the general case). The characteristic cohomology of the Milnor fiber $\mathcal{A}_{\mathcal{V}}(f_0; R)$, respectively of the complement $\mathcal{C}_{\mathcal{V}}(f_0; R)$ are subalgebras of the cohomology of the Milnor fibers, respectively the complement, with coefficients R in the corresponding cohomology. For a fixed $\mathcal{V}$, they are functorial over the category of singularities of type $\mathcal{V}$. In addition, for the link of $\mathcal{V}_0$ there is a characteristic cohomology subgroup $\mathcal{B}_{\mathcal{V}}(f_0, \mathbf{k})$ of the cohomology of the link over a field $\mathbf{k}$ of characteristic 0. The cohomologies $\mathcal{C}_{\mathcal{V}}(f_0; R)$ and $\mathcal{B}_{\mathcal{V}}(f_0, \mathbf{k})$ are shown to be invariant under the $\mathcal{K}_{\mathcal{V}}$-equivalence of defining germs f_0, and likewise $\mathcal{A}_{\mathcal{V}}(f_0; R)$ is shown to be invariant under the $\mathcal{K}_H$-equivalence of f_0 for H the defining equation of $\mathcal{V}, 0$.

We specialize to the case where $\mathcal{V}$ denotes any of the varieties of singular $m \times m$ complex matrices which may be either general, symmetric or skew-symmetric (with m even). For these varieties we have shown in another paper that their Milnor fibers and complements have compact "model submanifolds" for their homotopy types, which are classical symmetric spaces in the sense of Cartan. As a result, it follows that the characteristic subalgebras $\mathcal{A}_{\mathcal{V}}(f_0; R)$ and $\mathcal{C}_{\mathcal{V}}(f_0; R)$ are images of exterior algebras (or in one case a module on two generators over an exterior algebra). We extend these results to general $m \times p$ complex matrices.

We then give a geometric criterion involving "vanishing compact models" that detects when either of the functorial characteristic cohomologies contain a specific exterior subalgebra on ℓ generators and for the link that it contains an appropriate truncated and shifted version of the subalgebra. For matrix singularities we apply the geometric detection criterion by introducing a special type of "kite map germ of size ℓ"based on a given flag of subspaces. The general criterion which detects such nonvanishing characteristic cohomology is then given in terms of the defining germ f_0 containing such a kite map germ of size ℓ. Furthermore we use a restricted form of kite spaces to give a cohomological relation between the cohomology of local links and the global link.

Lastly we begin to indicate what form the cohomology of the Milnor fiber takes as a module over the characteristic subalgebra. We consider several other classes of examples, such as generic hyperplane and hypersurface arrangements, to begin to understand the form this decomposition takes.

Preliminary Version

Introduction

For a germ of a hypersurface $\mathcal{V}_0, 0 \subset \mathbb{C}^n, 0$ with a nonisolated singularity, a result of Kato-Masumoto [KM] states that the connectivity of the Milnor fiber may decrease by $r = \dim_{\mathbb{C}}\mathrm{sing}(\mathcal{V}_0)$. Thus, it may have nonzero (co)homology $H^j(\mathcal{V}_0)$ in dimension $n - 1 - r \leq j \leq n - 1$. For very low dimensional singular sets of dimension ≤ 2, with special forms for $\mathrm{sing}(\mathcal{V}_0)$ and the transverse types of the defining equation f_0 on $\mathrm{sing}(\mathcal{V}_0)$, the work of Siersma and coworkers Pellikan, Tibar, Nemethi, Zaharia, Van Straten, etc., have determined the topological structure of the Milnor fibers (see e.g. the survey [Si]). However, very little is known about the topology for hypersurfaces with higher dimensional singular sets. We consider in this paper how we may introduce in such a situation more information about the topology of a singularity $\mathcal{V}_0$, which is based on a "universal singularity" $\mathcal{V}$, even when it is highly nonisolated. This will be done by identifying how topological properties of $\mathcal{V}$ are inherited by $\mathcal{V}_0$.

We give a general formulation for the category of singularities $\mathcal{V}_0$ of "type $\mathcal{V}$" for a fixed germ of a variety $\mathcal{V}, 0 \subset \mathbb{C}^N, 0$ defined as $\mathcal{V}_0 = f_0^{-1}(\mathcal{V})$ for a germ $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^N, 0$ (which for a subcategory is transverse to $\mathcal{V}$ in an appropriate sense). If $\mathcal{V}$ is a hypersurface germ, then so is $\mathcal{V}_0$. If $\mathcal{V}, 0$ is a highly singular germ and $n > \operatorname{codim}_{\mathbb{C}}\operatorname{sing}(\mathcal{V})$, then by transversality, $\mathcal{V}_0, 0$ will also exhibit singularities of $\mathcal{V}$ up to codimension n, and hence also in general be highly singular. Nonetheless we define the characteristic cohomology for the Milnor fiber (for the hypersurface case), and the complement and link of $\mathcal{V}_0$ (in the general case).

The "characteristic cohomology algebra" of the Milnor fiber of $\mathcal{V}_0$ is defined as $\mathcal{A}_{\mathcal{V}}(f_0; R) = \widetilde{f_0}^*(H^*(F_w; R)$, for $\widetilde{f_0}; \mathcal{V}_w \to F_w$ the induced map of Milnor fibers. Likewise, the "characteristic cohomology algebra" of the link is defined to be $\mathcal{C}_{\mathcal{V}}(f_0; R) = f_0^*(H^*(\mathbb{C}^N \backslash \mathcal{V}; R)$ (which is understood in the sense of local cohomology). Both of these are shown to be well-defined and functorial over the category of singularities of type $\mathcal{V}$ for a fixed singularity $\mathcal{V}$. For a field $\mathbf{k}$ of characteristic 0, the "characteristic cohomology (subspace)" of the link, $\mathcal{B}_{\mathcal{V}}(f_0; \mathbf{k})$ is defined to be the Alexander dual of the Kronecker dual of $\mathcal{C}_{\mathcal{V}}(f_0; \mathbf{k})$. It is not functorial, but is natural with respect to a relative form of the Gysin homomorphism.

We show that $\mathcal{A}_{\mathcal{V}}(f_0; R)$ is invariant, up to an algebra isomorphism of the cohomology of the Milnor fiber, under $\mathcal{K}_H$-equivalence of f_0 (i.e. $\mathcal{K}$-equivalence of f_0 preserving the defining equation H of $\mathcal{V}$, see e.g. [DM]). Also, both $\mathcal{C}_{\mathcal{V}}(f_0; R)$ and $\mathcal{B}_{\mathcal{V}}(f_0; \mathbf{k})$ are invariant under $\mathcal{K}_{\mathcal{V}}$-equivalence of f_0, up to an algebra isomorphism of the cohomology of the complement, resp. the isomorphism of the cohomology group of the link. This will allow us to give a structural form for the cohomology of the Milnor fiber (in the hypersurface case) and of the complement (for general $\mathcal{V}$), as modules over corresponding "characteristic subalgebras". Furthermore, we give results about the exact form of these characteristic subalgebras.

We first specialize to the case where $\mathcal{V}$ denotes any of the varieties of singular $m \times m$ complex matrices which may be either general, symmetric or skew-symmetric (with m even). These give rise to "matrix singularities" $\mathcal{V}_0$ of any of the three types. For matrix singularities the characteristic cohomology will give the analogue of characteristic classes for vector bundles (as e.g. [MS]). For comparison, a vector bundle $E \to X$ over CW complex X is given by map $f_0 : X \to BG$ for G the structure group of E (e.g. O_n, U_n, Sp_n, SO_n, etc.). It is well-defined up to isomorphism by the homotopy class of f_0. Moreover the generators of $H^*(BG; R)$, for appropriate coefficient ring R pull-back via f_0^* to give the characteristic classes of E; so they generate a characteristic subalgebra of $H^*(X; R)$. The nonvanishing of the characteristic classes which then give various properties of E. Various polynomials in the classes correspond to Schubert cycles in the appropriate classifying spaces.

We will give analogous results for categories of matrix singularities of the various types. Homotopy invariance is replaced by invariance under the actions of the groups of diffeomorphisms $\mathcal{K}_H$ or $\mathcal{K}_{\mathcal{V}}$. For these varieties we have shown in another paper [D3] that they have compact "model submanifolds" for the homotopy types of both the Milnor fibers and the complements and these are classical symmetric spaces in the sense of Cartan. As a result, it will follow that the characteristic subalgebra is the image of an exterior algebra (or in one case a module on two generators over an exterior algebra) on an explicit set of generators.

We give a "detection criterion" for identifying in the characteristic sublgebra an exterior subalgebra on pull-backs of ℓ specific generators of the cohomology of the corresponding symmetric space. It is detected by the defining germ f_0 containing a special type of "unfurled kite map" of size ℓ. This will be valid for the Milnor fiber, complement, and link.

We will do this by using the support of appropriate exterior subalgebras of the Milnor fiber cohomology or of the complement cohomology for the varieties of singular matrices. This is done using results of [D4] giving the Schubert decomposition for the Milnor fiber and the complement to define "vanishing compact models" detecting these subalgebras. In §§5 and 9 we use the Schubert decompositions to exhibit vanishing compact models in the Milnor fibers and complements. Then, we give a detection criterion for exterior subalgebras of the characteristic cohomology using a class of "unfurled kite maps". Matrix singularities $\mathcal{V}_0, 0$ defined by germs f_0 which contain such an "unfurled kite map", are shown to have such subalgebras in their cohomology of Milnor fibers or complements and subgroups in their link cohomology. In the case of general or skew-symmetric matrices, the results for the Milnor fibers and complements is valid for cohomology over $\mathbb{Z}$ (and hence any coefficient ring R); while for symmetric matrices, the results apply both for cohomology with coefficients in a field of characteristic zero or for $\mathbb{Z}/2\mathbb{Z}$-coefficients. In all three cases for a field of characteristic zero, cohomology subgroups are detected for the links which are above the middle dimensions.

Furthermore, we extend in §10 the results for complements and links for $m \times m$ matrices to general $m \times p$ matrices. This includes determining the form of the characteristic cohomology and giving a detection criterion using an appropriate form of kite spaces and mappings.

A restricted form of the kite spaces serve a further purpose in §11 for identifying how the cohomology of local links of strata in the varieties of singular matrices relate to the cohomology of the global links.

In §12 we begin to investigate how the cohomology of the Milnor fiber can be understood as a module over the characteristic subalgebra and the role that the topology of the singular Milnor fiber plays. This is further consider examples in §13 of generic hyperplane and hypersurface arrangements to see what form this module structure takes.

Lastly, we consider in §13 a number of general classes of nonisolated complex singularities which are of a given "universal type". These include discriminants of finitely determined (holomorphic) map germs; bifurcation sets for $\mathcal{G}$-equivalence where $\mathcal{G}$ is a geometric subgroup of $\mathcal{A}$ or $\mathcal{K}$ in the holomorphic category; generic hyperplane or hypersurface arrangements based on special central complex hyperplane arrangements, and determinantal arrangements arising from exceptional orbit varieties of prehomogeneous spaces (which includes matrix singularities). We consider how specific results for these examples reveal the role that characteristic cohomology is playing for these other case.

CONTENTS

1. Characteristic Cohomology of Singularities of type $\mathcal{V}$

We begin by considering singularities arising as nonlinear sections of some given "universal " singularity $\mathcal{V}, 0$. There are many fundamental examples of such universal singularities which are, in particular, hypersurface singularities including: reflection hyperplane arrangements, discriminants of stable map germs, bifurcation sets for the $\mathcal{G}$-versal unfoldings of germs for many different singularity equivalence groups $\mathcal{G}$ which are "geometric subgroups of $\mathcal{A}$ or $\mathcal{K}$" (see e.g. [D2] and papers cited therein), exceptional orbit hypersurfaces of prehomogeneous spaces [D4] which include both reductive groups, e.g. [BM], and solvable groups [DP2], [DP3], as well as specifically the varieties of singular $m \times m$ matrices which may be general, symmetric, or skew-symmetric (if m is even). There are also other classes of universal singularities which are not hypersurface singularities, such as bifurcation sets for certain $\mathcal{G}$-versal unfoldings and varieties of singular $n \times m$ matrices with $n \neq m$. Part of the results we state will also be applicable to the non-hypersurface cases.

Category of Singularities of Type $\mathcal{V}$.

We recall from [D5] that given a germ of an analytic set $\mathcal{V}, 0 \subset \mathbb{C}^N, 0$, a "nonlinear section" is given by a germ of a holomorphic map $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^N, 0$ (so that $f_0(\mathbb{C}^n) \not\subset \mathcal{V}$), where n may take any value (including allowing $n > N$). The associated singularity of type $\mathcal{V}$ is $\mathcal{V}_0 = f_0^{-1}(\mathcal{V})$.

$$
\begin{array}{ccccc}
 & & \mathbb{C}^n, 0 & \xrightarrow{f_0} & \mathbb{C}^N, 0 \\
(1.1) & & \uparrow & & \uparrow \\
f_0^{-1}(\mathcal{V}) & = & \mathcal{V}_0, 0 & \longrightarrow & \mathcal{V}, 0
\end{array}
$$

We consider the *category of singularities of type* $\mathcal{V}$. The objects are the singularities of type $\mathcal{V}$. Given two singularities of type $\mathcal{V}$: $\mathcal{V}_0$ defined by $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^N, 0$ and $\mathcal{W}_0$ defined by $g_0 : \mathbb{C}^s, 0 \to \mathbb{C}^N, 0$, a morphism $\psi : \mathcal{W}_0, 0 \to \mathcal{V}_0, 0$ is given by a germ $\tilde{\psi} : \mathbb{C}^s, 0 \to \mathbb{C}^n, 0$ such that $g_0 = f_0 \circ \tilde{\psi}$. Such singularities of type $\mathcal{V}$ and the corresponding morphisms between them give a category on which we will define the characteristic cohomology.

The basic equivalence for studying the ambient equivalence of such $\mathcal{V}_0$ is $\mathcal{K}_{\mathcal{V}}$-equivalence of the germs f_0, which is a form of $\mathcal{K}$-equivalence which preserves $\mathcal{V}$, see e.g. [D5] or [D2]. This equivalence applied to an $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^N, 0$ can be viewed as the action on the section $graph(f_0) : \mathbb{C}^n, 0 \to \mathbb{C}^n \times \mathbb{C}^N, 0$ of the trivial vector bundle on $\mathbb{C}^n$ with fiber $\mathbb{C}^N$. It acts via diffeomorphisms of the fibers preserving each copy of $\mathcal{V}$ and which holomorphically varies pointwise on $\mathbb{C}^n, 0$. As such it is a type of gauge group.

We also consider the defining equation $H : \mathbb{C}^N, 0 \to \mathbb{C}, 0$ for $\mathcal{V}$. There is a stronger $\mathcal{K}_H$-equivalence within $\mathcal{K}_V$-equivalence (see [DM] and [D1]) where the diffeomorphisms of $\mathbb{C}^n \times \mathbb{C}^N, (0, 0)$ preserve the defining map germ $H \circ pr_2 : \mathbb{C}^n \times \mathbb{C}^N, (0, 0) \to \mathbb{C}, 0$ for $\mathbb{C}^n \times \mathcal{V}, (0, 0)$, where pr_2 denotes projection onto the second factor $\mathbb{C}^N, 0$. These diffeomorphisms not only preserve $\mathbb{C}^n \times \mathcal{V}$, but also $\mathbb{C}^n \times F$ for F a Milnor fiber of $\mathcal{V}$.

We further consider a subcategory of singularities of type $\mathcal{V}$ where the germ f_0 is transverse to $\mathcal{V}$ on the complement of 0 in $\mathbb{C}^n$. Transversality can be either in a geometric sense of transversality to the canonical Whitney stratification of $\mathcal{V}$ or in an algebraic sense using the module of logarithmic vector fields (see [D1]) and these agree if $\mathcal{V}$ is holonomic in the sense of Saito [Sa]. In these cases the corresponding germ is finitely $\mathcal{K}_{\mathcal{V}}$-determined. These singularities and the corresponding morphisms between them give a subcategory of "finitely determined singularities of type $\mathcal{V}$".

In analyzing the topology of such singularities $\mathcal{V}_0$ there are three contributions:

a) the contribution from the topology of the germ f_0 and its geometric interaction with $\mathcal{V}$;
b) the contribution from the topology of $\mathcal{V}$;
c) the interaction between these two contributions combining to give the topology of $\mathcal{V}_0$.

For a), there have been results introduced for discriminants of finitely determined mappings and more generally finitely determined nonlinear sections of free divisors and complete intersections in [DM] and [D1], and of the varieties of $m \times m$ matrices in [GM] and [DP3], using a stabilization of the mapping to obtain a "singular Milnor fiber" homotopy equivalent to a bouquet of spheres, with the number of such spheres computed algebraically. However, this provides no information about b). The characteristic cohomology which we will introduce will specifically address b) and provide complementary information to that given for a). We briefly indicate in §12 how these two contributions combine for c).

Characteristic Cohomology on the Category of Singularities of Type $\mathcal{V}$.

We begin with the definition for the Milnor fiber in the case $\mathcal{V}, 0$ is a hypersurface singularity.

Characteristic Cohomology $\mathcal{A}_{\mathcal{V}}(f_0, R)$.

Let $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^N, 0$ define the singularity $\mathcal{V}_0$. For $\mathcal{V}$ there exists $0 < \eta << \delta$ such that for balls $B_\eta \subset \mathbb{C}$ and $B_\delta \subset \mathbb{C}^N$ (with all balls centered 0), we let $\mathcal{F}_\delta = H^{-1}(B^*_\eta) \cap B_\delta$ so $H : \mathcal{F}_\delta \to B^*_\eta$ is the Milnor fibration of H, with Milnor fiber $F_w = H^{-1}(w) \cap B_\delta$ for each $w \in B^*_\eta$. By continuity, there is an $\varepsilon > 0$ so that $f_0(B_\varepsilon) \subset \mathcal{F}_\delta$. By possibly shrinking all three values, $H \circ f_0 : f_0^{-1}(\mathcal{F}_\delta) \cap B_\varepsilon \to B^*_\eta$ is the Milnor fibration of $H \circ f_0$. Then, the Milnor fiber of $H \circ f_0$ for $w \in B^*_\eta$ is given by

$$\mathcal{V}_w \;=\; (H \circ f_0)^{-1}(w) \cap B_\varepsilon \;=\; f_0^{-1}(F_w) \cap B_\varepsilon \,.$$

Thus, if we denote $f_0|\mathcal{V}_w = f_{0,w}$, then in cohomology with coefficient ring R, $f^*_{0,w} : H^*(F_w; R) \to H^*(\mathcal{V}_w; R)$. we let

$$\mathcal{A}_{\mathcal{V}}(f_0; R) \;\overset{def}{=}\; f^*_{0,w}(H^*(F_w; R))\,, \tag{1.2}$$

We formally define the characteristic cohomology of the Milnor fiber.

Definition 1.1. Let $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^N, 0$ define $\mathcal{V}_0 = f_0^{-1}(\mathcal{V})$. We define the *characteristic cohomology subalgebra of the Milnor fiber* of $\mathcal{V}_0$, to be cohomology subalgebra of the Milnor fiber $H^*(\mathcal{V}_w; R)$ of $\mathcal{V}_0$ given by (1.2).

Independence of $\mathcal{A}_{\mathcal{V}}(f_0, R)$ on the Milnor Fiber under Cohomology Isomorphism.

Given another $w' \in B^*_\eta$, let $\gamma(t)$ denote a simple path in B^*_η from w to w'. We may first lift $\gamma(t)$ to an isotopy $\Phi_t : F_w \to F_{\gamma(t)}$ of the restriction of the Milnor fibration from F_w to $F_{w'}$. We can also lift $\gamma(t)$ to an isotopy $\Psi_t : \mathcal{V}_w \to \mathcal{V}_{\gamma(t)}$ of the restriction of the Milnor fibration from $\mathcal{V}_w$ to $\mathcal{V}_{w'}$. Then, $\Phi_t^{-1} \circ f_0 \circ \Psi_t : \mathcal{V}_w \to F_w$ defines a homotopy from $f_{0,w}$ to $\Phi_1^{-1} \circ f_{0,w'} \circ \Psi_1$. Thus, $f^*_{0,w} = \Psi_1^* \circ f^*_{0,w'} \circ \Phi_1^{*\,-1}$. Then, $\Phi_1^{*\,-1} : H^*(F_{w'}; R) \simeq H^*(F_w; R)$, and $\Psi_1^* : H^*(\mathcal{V}_{w'}; R) \simeq H^*(\mathcal{V}_w; R)$. Hence, $f^*_{0,w'}(H^*(\mathcal{V}_{w'}; R))$ is mapped under the cohomology algebra isomorphism Ψ_1^* to $f^*_{0,w}(H^*(\mathcal{V}_w; R))$. Thus, Ψ_1^* maps the characteristic cohomology for the Milnor fiber of $\mathcal{V}_w$ to that of $\mathcal{V}_{w'}$.

We also remark that if we consider a second set of values $0 < \eta' < \eta$, $0 < \delta' < \delta$, and $0 < \varepsilon' < \varepsilon$ for the Milnor fibers of H and $H \circ f_0$, and choose $w \in B^*_{\eta'}$ so that the Minor fiber $\mathcal{V}'_w$ is transverse to the spheres $S^{2n-1}_{\varepsilon''}$ for $\varepsilon' < \varepsilon'' < \varepsilon$, then $i_w : \mathcal{V}'_w \subset \mathcal{V}_w$ is a homotopy equivalence so the characteristic cohomology for $\mathcal{V}'_w$ is mapped isomorphically to that of $\mathcal{V}_w$. Hence, the characteristic cohomology is well-defined independent of the Milnor fiber up to Milnor fiber cohomology isomorphism. When we want to refer to the characteristic cohomology at more than one point $w \in B^*_\eta$, we use the notation $\mathcal{A}(f_0, R)_w$ to denote the representative in the Milnor fiber cohomology $H^*(\mathcal{V}_w; R)$.

Remark 1.2. We consider two consequences of the above arguments. First, if we choose a convex neighborhood $w \in U \subset B^*_\eta$, then as the paths in U between w and any other w' are homotopic, it follows that the induced diffeomorphisms between the Milnor fibers $\mathcal{V}_w$ and $\mathcal{V}_{w'}$, resp. F_w and $F_{w'}$, are homotopic so the algebra isomorphisms between the cohomology of the Milnor fibers over U is well-defined. This gives a local trivialization of the unions $\cup_{w' \in U} \mathcal{A}(f_0, R)_{w'}$, resp. $\cup_{w' \in U} H^*(\mathcal{V}_{w'}; R)$.

On overlaps of two such neighborhoods the transition isomorphisms are constant. Together they give a locally constant system on B^*_η. Second, if $\gamma(t)$ is a simple loop in B^*_η from w around 0, then the preceding arguments show the monodromy will map the characteristic cohomology to itself. Thus, the characteristic cohomology inherits two properties from the Milnor fiber cohomology. In this paper we will not attempt to make use of these additional properties.

Characteristic Cohomology $\mathcal{C}_{\mathcal{V}}(f_0, R)$.

We next introduce the characteristic cohomology of the complement of $\mathcal{V}_0$ in the case where $\mathcal{V}, 0$ need not be a hypersurface singularity. This proceeds somewhat analogously to the case of Milnor fibers. Let $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^N, 0$ define $\mathcal{V}_0 = f_0^{-1}(\mathcal{V})$. Then, we consider a representative $\tilde{f}_0 : U \to W$ for which $\mathcal{V}$ also has a representative on W, and we still denote the representative by $\mathcal{V}$. Then, $\tilde{f}_0^{-1}(\mathcal{V})$ is a representative for $\mathcal{V}_0$ which we still denote by $\mathcal{V}_0$. Then, by stratification theory (see e.g. Mather [M1], [M2] or Gibson et al [GDW]), there are $0 < \delta_0, \varepsilon_0$ so that for $0 < \delta' < \delta \leq \delta_0$ and $0 < \varepsilon' < \varepsilon \leq \varepsilon_0$:

i) $\overline{B}_{\delta_0} \subset W$ and $\overline{B}_{\varepsilon_0} \subset U$,
ii) $\partial\overline{B}_\delta$ is transverse to $\mathcal{V}$ and $\partial\overline{B}_\varepsilon$ is transverse to $\mathcal{V}_0$.
iii) $\mathcal{V}_0 \cap \overline{B}_{\varepsilon'}$ is ambiently homeomorphic to the cone on $\mathcal{V}_0 \cap \partial\overline{B}_{\varepsilon'}$, as is $\mathcal{V} \cap \overline{B}_{\delta'}$ ambiently homeomorphic to the cone on $\mathcal{V} \cap \partial\overline{B}_{\delta'}$, and
iv) the inclusions of pairs
$$(\overline{B}_{\varepsilon'}, \mathcal{V}_0 \cap \overline{B}_{\varepsilon'}) \hookrightarrow (\overline{B}_\varepsilon, \mathcal{V}_0 \cap \overline{B}_\varepsilon)$$
and
$$(\overline{B}_{\delta'}, \mathcal{V} \cap \overline{B}_{\delta'}) \hookrightarrow (\overline{B}_\delta, \mathcal{V} \cap \overline{B}_\delta)\,,$$
are homotopy equivalences.

Thus, if $f_0(\overline{B}_{\varepsilon'}) \subset B_{\delta'}$, and $f_0(\overline{B}_\varepsilon) \subset B_\delta$, then there is the commutative diagram

$$\begin{array}{ccc} H^*(\overline{B}_\delta \backslash \mathcal{V}; R) & \xrightarrow{f_0^*} & H^*(\overline{B}_\varepsilon \backslash \mathcal{V}_0; R) \\ \simeq\downarrow & & \simeq\downarrow \\ H^*(\overline{B}_{\delta'} \backslash \mathcal{V}; R) & \xrightarrow{f_0^*} & H^*(\overline{B}_{\varepsilon'} \backslash \mathcal{V}_0; R) \end{array} \tag{1.3}$$

and the vertical maps are isomorphisms by property (iv). Thus, via the vertical isomorphisms, the induced homomorphisms $f_0^* : H^*(\overline{B}_\delta \backslash \mathcal{V}; R) \to H^*(\overline{B}_\varepsilon \backslash \mathcal{V}_0; R)$ are independent of $0 < \varepsilon < \varepsilon_0$ and $0 < \delta < \delta_0$. Hence, the induced isomorphisms $f_0^*(H^*(\overline{B}_\delta \backslash \mathcal{V}; R)) \simeq f_0^*(H^*(\overline{B}_{\delta'} \backslash \mathcal{V}; R))$ yield an inverse system with limit isomorphic to each of these groups, giving a well-defined cohomology subalgebra.

Definition 1.3. Let $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^N, 0$ define $\mathcal{V}_0 = f_0^{-1}(\mathcal{V})$. We define the *characteristic cohomology (algebra) of the complement* of $\mathcal{V}_0$, to be cohomology subalgebra which is the direct limit
$$\mathcal{C}_{\mathcal{V}}(f_0, R) \stackrel{def}{=} \lim_{\rightarrow} f_0^*(H^*(\overline{B}_\delta \backslash \mathcal{V}; R))\,.$$

We note that this cohomology is really in local cohomology of the complement, but it is given by the complement in sufficient small neighborhoods.

Just as for complements, singularities $\mathcal{V}_0$ of type $\mathcal{V}$ also have characteristic cohomology for the link.

Characteristic Cohomology $\mathcal{B}_{\mathcal{V}}(f_0, R)$.

We use the same notation as above for the complement where again $\mathcal{V}, 0$ need not be a hypersurface singularity. In this case, we consider $R = \mathbf{k}$, a field of characteristic 0. By the conical structure for the pair $(\overline{B}_\varepsilon, \overline{B}_\varepsilon \cap \mathcal{V}_0)$, it follows that the inclusion $j_\varepsilon : (S^{2n-1}_\varepsilon \backslash \mathcal{V}_0) \subset \overline{B}_\varepsilon \backslash \mathcal{V}_0$ is a homotopy equivalence. Thus, $j^*_\varepsilon : H^*(\overline{B}_\varepsilon \backslash \mathcal{V}_0; \mathbf{k}) \simeq H^*(S^{2n-1}_\varepsilon \backslash \mathcal{V}_0; \mathbf{k})$ is an isomorphism.

For each $0 < \varepsilon \leq \varepsilon_0$, there is the Kronecker dual graded subgroup of

$$j^*_\varepsilon \circ f^*_0(H^*(\overline{B}_\delta \backslash \mathcal{V}; \mathbf{k})) \ \subset \ H^*(S^{2n-1}_\varepsilon \backslash \mathcal{V}_0; \mathbf{k}),$$

which we denote by $\Gamma_{\mathcal{V}}(f_0; \mathbf{k}) \subset H_*(S^{2n-1}_\varepsilon \backslash \mathcal{V}_0; \mathbf{k})$. We note that for the Kronecker pairing we may choose a dual basis for $H_*(\overline{B}_\varepsilon \backslash \mathcal{V}_0; \mathbf{k})$ that extends a basis for $j^*_\varepsilon \circ f^*_0(H^*(\overline{B}_\delta \backslash \mathcal{V}; R))$, so it is dually paired to $\Gamma_{\mathcal{V}}(f_0; \mathbf{k})$.

Then, we can apply a form of Alexander duality for subspaces of spheres, [Ma, Chap. XIV, Thm 6.6] or see e.g. [D3, Prop. 1.9]. For $L(\mathcal{V}_0) = S^{2n-1}_\varepsilon \cap \mathcal{V}_0$, the link of $\mathcal{V}_0$,

$$\alpha : \widetilde{H}^j(L(\mathcal{V}_0); \mathbf{k}) \simeq \widetilde{H}_{2n-2-j}(S^{2n-1}_\varepsilon \backslash L(\mathcal{V}_0); \mathbf{k}) \qquad \text{for all j} \tag{1.4}$$

Then, if $\widetilde{\Gamma}_{\mathcal{V}}(f_0; \mathbf{k})$ denotes the corresponding reduced homology obtained by removing H_0 from $\Gamma_{\mathcal{V}}(f_0; \mathbf{k})$, then we define the characteristic cohomology for the link.

Definition 1.4. Let $f_0 : \mathbb{C}^n, 0 \to M, 0$ define $\mathcal{V}_0 = f_0^{-1}(\mathcal{V})$. We define the *characteristic cohomology of the link* of $\mathcal{V}_0$, to be

$$\mathcal{B}_{\mathcal{V}}(f_0; \mathbf{k}) \ \stackrel{def}{=} \ \alpha^{-1}(\widetilde{\Gamma}_{\mathcal{V}}(f_0); \mathbf{k}) \tag{1.5}$$

Since the definition in (1.5) is independent, up to isomorphism, of ε, this gives a well-defined graded cohomology subgroup in the cohomology of the link. However, because of the use of Alexander duality, this is not a subalgebra as is the case for the Milnor fiber and the complement. Also, the actual subgroup does depend upon the choice of basis for the Kronecker pairing; however, we still obtain subspaces in each degree whose dimensions are independent of choices.

Remark 1.5. On first glance it might seem that it would be more natural to define the characteristic cohomology of the link to be

$$\mathcal{L}_{\mathcal{V}}(f_0, R) \ \stackrel{def}{=} \ \lim_{\rightarrow} f^*_0(H^*(\overline{B}_\delta \cap \mathcal{V}; R)).$$

However, we shall see in §9 that this subgroup of the cohomology of the link does not capture the directly identifiable cohomology in $H^*(L(\mathcal{V}_0), R)$. Specifically this cohomology will lie above the middle dimension, while theorems such as the Le-Hamm Local Lefschetz Theorem, see e.g. [HL] or [GMc, Part 2, §1.2, Thm 1], when they are applicable only concern dimensions below the middle dimension.

Functoriality of Characteristic Cohomology $\mathcal{A}_{\mathcal{V}}(f_0, R)$ and $\mathcal{C}_{\mathcal{V}}(f_0, R)$.

We complete this section by establishing the functoriality of both $\mathcal{A}_{\mathcal{V}}(f_0, R)$ and $\mathcal{C}_{\mathcal{V}}(f_0, R)$ on the category of singularities of type $\mathcal{V}$.

Lemma 1.6. *Given $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^N, 0$ defining $\mathcal{V}, 0$ and $g_0 : \mathbb{C}^s, 0 \to \mathbb{C}^N, 0$ defining $\mathcal{W}, 0$ both of type $\mathcal{V}, 0 \subset \mathbb{C}^N, 0$ with a morphism $\varphi : \mathcal{W}_0, 0 \to \mathcal{V}_0, 0$ defined by $\tilde{\varphi} : \mathbb{C}^s, 0 \to \mathbb{C}^n, 0$. Then, φ induces the algebra homomorphisms $\tilde{\varphi}^* : \mathcal{A}_{\mathcal{V}}(f_0, R) \to \mathcal{A}_{\mathcal{V}}(g_0, R)$ and $\tilde{\varphi}^* : \mathcal{C}_{\mathcal{V}}(f_0, R) \to \mathcal{C}_{\mathcal{V}}(g_0, R)$. Moreover, both $\mathcal{A}_{\mathcal{V}}(f_0, R)$ and $\mathcal{C}_{\mathcal{V}}(f_0, R)$ are functorial.*

Then, we shall let $\varphi^* : \mathcal{A}_{\mathcal{V}}(f_0, R) \to \mathcal{A}_{\mathcal{V}}(g_0, R)$ and $\varphi^* : \mathcal{C}_{\mathcal{V}}(f_0, R) \to \mathcal{C}_{\mathcal{V}}(g_0, R)$ denote the induced algebra homomorphisms defined by $\tilde{\varphi}^*$.

Proof. We begin by showing that $\tilde{\phi}^*$ gives a well-defined homomorphism between the algebras in each case. We consider $0 < \eta << \varepsilon_2, \varepsilon_1, \delta$ so that:

i) $\varphi(B_{\varepsilon_2}) \subset B_{\varepsilon_1}$, $f_0(B_{\varepsilon_1}) \subset B_\delta$, and $H(B_\delta) \subset B_\eta$; and
ii) $H : H^{-1}(B^*_\eta) \cap B_\delta \to B^*_\eta$ is the Milnor fibration for H; $H \circ f_0 : (H \circ f_0)^{-1}(B^*_\eta) \cap B_{\varepsilon_1} \cap \to B^*_\eta$ is the Milnor fibration for $H \circ f_0$; and $H \circ f_0 \circ \tilde{\varphi} : (H \circ f_0 \circ \tilde{\varphi})^{-1}(B^*_\eta) \cap B_{\varepsilon_2} \cap \to B^*_\eta$ is the Milnor fibration for $H \circ f_0 \circ \tilde{\varphi} = H \circ g_0$.

Then, for $w \in B^*_\eta$ we have the induced maps for the cohomology of the Milnor fibers

$$H^*(F_w; R) \xrightarrow{f_0^*} H^*(\mathcal{V}_w; R) \xrightarrow{\tilde{\varphi}^*} H^*(\mathcal{S}_w; R)\,. \tag{1.6}$$

Then the composition in (1.6) is

$$H^*(F_w; R) \xrightarrow{\tilde{\varphi}^*_w \circ f^*_{0\,w}} H^*(\mathcal{S}_w; R)\,. \tag{1.7}$$

The image of this composition in (1.7) defines $\mathcal{A}_{\mathcal{V}}(g_0, R)$ and factors through (1.8). Hence, $\tilde{\varphi}^*_w$ induces a well-defined map $\tilde{\varphi}^* : \mathcal{A}_{\mathcal{V}}(f_0, R) \to \mathcal{A}_{\mathcal{V}}(g_0, R)$.

$$H^*(F_w; R) \xrightarrow{f^*_{0\,w}} H^*(\mathcal{V}_w; R)\,. \tag{1.8}$$

For functoriality, we include a third singularity $\mathcal{Z}_0$ of type $\mathcal{V}$ given by $h_0 : \mathbb{C}^r, 0 \to \mathbb{C}^N, 0$ such that there is a map germ $\tilde{\psi} : \mathbb{C}^r, 0 \to \mathbb{C}^s, 0$ so that $g_0 \circ \tilde{\psi} = h_0$. Then, choosing an additional $0 < \eta << \varepsilon_3$ so that $\tilde{\psi}(B_{\varepsilon_3}) \subset B_{\varepsilon_2}$, and $H \circ f_0 \circ \tilde{\varphi} \circ \tilde{\psi} : (H \circ f_0 \circ \tilde{\varphi} \circ \tilde{\psi})^{-1}(B^*_\eta) \cap B_{\varepsilon_3} \cap \to B^*_\eta$ is the Milnor fibration for $H \circ f_0 \circ \tilde{\varphi} \circ \tilde{\psi} = H \circ h_0$. Then, by functoriality in cohomology, $\tilde{\psi}$ maps the image in (1.7) to $H^*(\mathcal{Z}_w; R)$, for $\mathcal{Z}_w$ the Milnor fiber of $H \circ h_0$ over w, and $(\tilde{\varphi} \circ \tilde{\psi})^* = \tilde{\psi}^* \circ \tilde{\varphi}^*$. Hence, using our notation for the induced maps on characteristic cohomology, $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$.

For $\mathcal{C}_{\mathcal{V}}(f_0, R)$, $\mathcal{C}_{\mathcal{V}}(g_0, R)$, and $\mathcal{C}_{\mathcal{V}}(h_0, R)$, the proof is similar, except we replace the Milnor fibers by the complements $B_\delta \backslash \mathcal{V}$, resp. $B_{\varepsilon_1} \backslash \mathcal{V}_0$, resp. $B_{\varepsilon_2} \backslash \mathcal{W}_0$, resp. $B_{\varepsilon_3} \backslash \mathcal{Z}_0$ and consider the induced maps in cohomology of these complements by f_0^*, resp. $\tilde{\varphi}^*$, resp. $\tilde{\psi}^*$ and their compositions. ◻

One immediate consequence of functoriality is the detection of the nonvanishing characteristic cohomology. We note for the identity map $id : \mathbb{C}^N, 0 \to \mathbb{C}^N, 0$, $\mathcal{A}_{\mathcal{V}}(id, R)_w = H^*(F_w; R)$. With the above notation for a morphism $\varphi : \mathcal{W}_0, 0 \to \mathcal{V}_0, 0$ defined by $\tilde{\varphi} : \mathbb{C}^p, 0 \to \mathbb{C}^n, 0$ with $\mathcal{V}_0, 0$ defined by $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^N, 0$ and $\mathcal{W}_0, 0$ defined by $g_0 : \mathbb{C}^p, 0 \to \mathbb{C}^N, 0$. Then, we have the corollary.

Corollary 1.7. *If $g_0^* : \mathcal{A}_{\mathcal{V}}(id, R) \to \mathcal{A}_{\mathcal{V}}(g_0, R)$ induces an isomorphism from a graded subgroup $E \subset \mathcal{A}_{\mathcal{V}}(id, R)$ to a subgroup of $\mathcal{A}_{\mathcal{V}}(g_0, R)$, then f_0^* induces an isomorphism from E to a subgroup of $\mathcal{A}_{\mathcal{V}}(f_0, R)$.*

There is an analogous result for $\mathcal{C}_{\mathcal{V}}(f_0, R)$ and the complement.

Proof. By functoriality, we have for the sequence

$$\mathcal{A}_{\mathcal{V}}(id, R) \xrightarrow{f_0^*} \mathcal{A}_{\mathcal{V}}(g_0, R) \xrightarrow{\varphi^*} \mathcal{A}_{\mathcal{V}}(g_0, R)$$

the composition is $\varphi^* \circ f_0^* = g_0^*$. As g_0^* maps E isomorphically to its image, so must f_0^* map E isomorphically to its image. ◻

We will see how we can apply this idea in §3 with applications for matrix singularities given beginning in §refS:sec5.

Remark 1.8. Although $\mathcal{B}_{\mathcal{V}}(f_0, R)$ is not functorial, it does satisfy a relation involving a type of relative Gysin homomorphism, where in place of Poincare duality, Alexander duality is used because the links are not manifolds. For a morphism $\varphi : \mathcal{W}_0, 0 \to \mathcal{V}_0, 0$ defined by $\tilde{\varphi}$ we have a map for sufficiently small $0 < \eta << \varepsilon_2, \varepsilon 1$ so that $\tilde{\varphi}(\overline{B}_{\varepsilon_2}) \subset B_{\varepsilon_1}$ and $f_0(\overline{B}_{\varepsilon_1}) \subset B_\delta$. Then,

$$\widetilde{H}^j(S^{2s-1}_{\varepsilon_2} \cap \mathcal{W}_0; \mathbf{k}) \overset{\alpha}{\simeq} \widetilde{H}_{2s-2-j}(S^{2s-1}_{\varepsilon_2} \backslash \mathcal{W}_0; \mathbf{k}) \xrightarrow{j_{\varepsilon_2 *}} \widetilde{H}_{2s-2-j}(B_{\varepsilon_2} \backslash \mathcal{W}_0; \mathbf{k}) \xrightarrow{\tilde{\varphi}_*}$$

(1.9)

$$\widetilde{H}_{2s-2-j}(B_{\varepsilon_1} \backslash \mathcal{V}_0; \mathbf{k}) \simeq \widetilde{H}_{2s-2-j}(S^{2n-1}_{\varepsilon_1} \backslash \mathcal{V}_0; \mathbf{k}) \overset{\alpha^{-1}}{\simeq} \widetilde{H}^{2(n-s)+j}(S^{2n-2}_{\varepsilon_1} \cap \mathcal{V}_0; \mathbf{k})$$

The composition gives a homomorphism $\widetilde{H}^j(S^{2s-1}_{\varepsilon_2} \cap \mathcal{W}_0; \mathbf{k}) \to \widetilde{H}^{2(n-s)+j}(S^{2n-2}_{\varepsilon_1} \cap \mathcal{V}_0; \mathbf{k})$. Then, via the identification for different ε_i, we obtain a form of Gysin homomorphism

$$\varphi_* : \widetilde{H}^j(L(\mathcal{W}_0); \mathbf{k}) \longrightarrow \widetilde{H}^{2(n-s)+j}(L(\mathcal{V}_0); \mathbf{k})\,. \tag{1.10}$$

Also, by choosing consistent bases for the cohomology, this will induce a Gysin-type homomorphism $\mathcal{B}_{\mathcal{W}}(g_0; \mathbf{k}) \to \mathcal{B}_{\mathcal{V}}(f_0; \mathbf{k})$, which shifts degrees by $2(n-s)$.

2. $\mathcal{K}_H$ and $\mathcal{K}_{\mathcal{V}}$ Invariance of Characteristic Cohomology

We next turn to the invariance properties of the characteristic cohomology.

Invariance of Characteristic Cohomolgy $\mathcal{A}_{\mathcal{V}}(f_0; R)$ under $\mathcal{K}_H$ Equivalence.

The dependence of $\mathcal{A}_{\mathcal{V}}(f_0; R)$ on f_0 is clarified by the next proposition.

Proposition 2.1. *Suppose $f_i : \mathbb{C}^n, 0 \to \mathbb{C}^N, 0$, $i = 1, 2$ are $\mathcal{K}_H$–equivalent. Let F_i, $i = 1, 2$, denote the Milnor fibers of $H \circ f_i$ for a $w \in B^*_\eta$. Then, for any coefficient ring R, there is a cohomology algebra isomorphism $\alpha : H^*(F_1; R) \simeq H^*(F_2; R)$ such that $\alpha(\mathcal{A}_{\mathcal{V}}(f_1; R)) = \mathcal{A}_{\mathcal{V}}(f_2; R)$.*

Hence, the structure of the cohomology of the Milnor fiber of $H \circ f_0$ as a graded algebra (or graded module) over $\mathcal{A}_{\mathcal{V}}(f_1; R)$ is, up to isomorphism, independent of the $\mathcal{K}_H$–equivalence class of f_0.

.

Proof of Proposition 2.1. By the $\mathcal{K}_H$–equivalence of the germ $f_i : \mathbb{C}^n, 0 \to \mathbb{C}^N, 0$, there are representatives $f_i : U \to W$, for open neighborhoods U and W, and a diffeomorphism onto a subspace

$$\begin{aligned} \Phi : U' \times W' &\to U \times W \\ (x, y) \quad &\mapsto \quad (\varphi(x), \varphi_1(x, y)) \end{aligned} \tag{2.1}$$

sending $(0, 0) \mapsto (0, 0)$ such that Φ preserves $H \circ pr_2$ for $pr_2 : \mathbb{C}^n \times \mathbb{C}^N$ the projection onto the second factor, and so that $f_2(\varphi(x)) = \varphi_1(x, f_1(x))$ for all $x \in U'$. Thus, $\Phi(\text{graph}(f_1)) = \text{graph}(f_2) \cap Im(\Phi)$, and for any Milnor fiber F_w of H, $\Phi(\mathbb{C}^n \times F_w) = (\mathbb{C}^n \times F_w) \cap Im(\Phi)$.

We let $g_i = H \circ f_i$. Next we choose $0 < \eta_1 << \delta_1 << \varepsilon_1$ so that

i) $B_{\delta_1} \subset W'$;

ii) $B_{\varepsilon_1} \subset U'$;
iii) the Milnor fibration of H is given by $H : H^{-1}(B^*_{\eta_1}) \cap B_{\delta_1} \to B^*_{\eta_1}$; and
iv) the Milnor fibration of each g_i is given by $g_i : g_i^{-1}(B^*_{\eta_1}) \cap B_{\varepsilon_1} \to B^*_{\eta_1}$.

Next, we begin to find a series of $(\eta_j, \delta_j, \varepsilon_j)$ so that:

1) $0 < \eta_{j+1} < \eta_j$; $0 < \delta_{j+1} < \delta_j$, and $0 < \varepsilon_{j+1} < \varepsilon_j$ and

$$g_i : g_i^{-1}(B^*_{\eta_{j+1}}) \cap B_{\varepsilon_{j+1}} \to B^*_{\eta_{j+1}} \quad \text{and} \quad H : H^{-1}(B^*_{\eta_{j+1}}) \cap B_{\delta_{j+1}} \to B^*_{\eta_{j+1}} \tag{2.2}$$

are Milnor fibrations for g_i, $i = 1, 2$, resp. H.

2)

$$T_{j+1} \overset{def}{=} \Phi(B_{\varepsilon_{j+1}} \times B_{\delta_{j+1}}) \subset B_{\varepsilon_j} \times B_{\delta_j} . \tag{2.3}$$

3) $(B_{\varepsilon_{j+1}} \times B_{\delta_{j+1}}) \subset T_j$ (as T_j is an open neighborhood of $(0, 0)$).
4) If $\mathcal{V}_w^{(i,j)}$ denotes the Milnor fiber of $g_i : g_i^{-1}(B^*_{\eta_j}) \cap B_{\varepsilon_j} \to B^*_{\eta_j}$, then for $w \in B^*_{\eta_{j+1}}$ the inclusions of Milnor fibers in (2.4) are homotopy equivalences.

$$\mathcal{V}_w^{(i,j+1)} \subset \mathcal{V}_w^{(i,j)} ; \tag{2.4}$$

5) We repeat these steps for $j = 1, \dots, 4$.

We observe that as both Φ and the graph maps are diffeomorphisms, $\Phi : \mathcal{V}_w^{(i,j)} \simeq \Phi(\text{graph}(\mathcal{V}_w^{(i,j)}))$. We choose a $w \in B^*_{\eta_4}$ and let $Y_j = \text{graph}(\mathcal{V}_w^{(2,j)})$ and $Z_j = \Phi(\text{graph}(\mathcal{V}_w^{(1,j)}))$. Consider the sequence of inclusions and mapping

$$Z_4 \subset Y_3 \subset Z_2 \subset Y_1 \overset{H}{\to} (H^{-1}(w) \cap B_{\eta_4}) \tag{2.5}$$

Then, for cohomology (with coefficients in R understood)

$$H^*(H^{-1}(w) \cap B_{\eta_4}) \overset{H^*}{\to} H^*(Y_1) \to H^*(Z_2) \to H^*(Y_3) \to H^*(Z_4) \tag{2.6}$$

Now the composition $H^*(Y_1) \to H^*(Z_2) \to H^*(Y_3)$ is an isomorphism; hence $H^*(Z_2) \to H^*(Y_3)$ is surjective. Second, the composition $H^*(Z_2) \to H^*(Y_3) \to H^*(Z_4)$ is also an isomorphism so $H^*(Z_2) \to H^*(Y_3)$ is one-one. Thus, $H^*(Z_2) \to H^*(Y_3)$ is an isomorphism. Hence, so are the other inclusions isomorphisms.

A similar argument for the Milnor fibers of H for the various j, together with Φ preserving $H^{-1}(w)$ implies that Φ^* induces an isomorphism of the cohomology of the Milnor fiber. Since the map $\Phi : \text{graph}(\mathcal{V}_w^{(1,2)}) \to \text{graph}(\mathcal{V}_w^{(2,1)})$ commutes with H, we deduce that the induced isomorphism from Φ^* preserves the subalgebra $pr_2^*(H^*(H^{-1}(w)) \cap B_\delta)$. By the isomorphism on cohomology via graph*, we obtain the preservation of the characteristic subalgebra. □

Remark 2.2. We can apply the preceding argument for the sequence of inclusions in (2.5) conclude $\Phi : \text{graph}(\mathcal{V}_w^{(1,2)}) \to \text{graph}(\mathcal{V}_w^{(2,1)})$ induces an isomorphism for both integer homology and the fundamental group. As the closures of both of these spaces are smooth manifolds with boundaries and hence have CW-complex structures, it follows by the Hurewicz theorem that Φ is a homotopy equivalence.

Invariance of Characteristic Cohomology $\mathcal{C}_{\mathcal{V}}(f_0; R)$ and $\mathcal{C}_{\mathcal{B}}(f_0; R)$ under $\mathcal{K}_{\mathcal{V}}$ Equivalence.

In analogy with Proposition 2.1, the dependence of $\mathcal{C}_{\mathcal{V}}(f_0; R)$ and $\mathcal{B}_{\mathcal{V}}(f_0; R)$ on the $\mathcal{K}_{\mathcal{V}}$-equivalence class of f_0 is given by the next proposition.

Proposition 2.3. *Suppose $f_i : \mathbb{C}^n, 0 \to \mathbb{C}^N, 0$, $i = 1, 2$ are $\mathcal{K}_{\mathcal{V}}$–equivalent. Let $\mathcal{V}_i = f_i^{-1}(\mathcal{V})$. Then, for any coefficient ring R, there is a cohomology algebra isomorphism $\beta : H^*(\mathbb{C}^n\backslash\mathcal{V}_1; R) \simeq H^*(\mathbb{C}^n\backslash\mathcal{V}_2; R)$ such that $\beta(\mathcal{C}_{\mathcal{V}}(f_1; R)) = \mathcal{C}_{\mathcal{V}}(f_2; R)$.*

Hence, the structure of the cohomology of the complement $\mathbb{C}^n\backslash\mathcal{V}_i$ as a graded algebra (or graded module) over $\mathcal{C}_{\mathcal{V}}(f_1; R)$ is, up to isomorphism, independent of the $\mathcal{K}_{\mathcal{V}}$–equivalence class of f_i.

Proof. The proof is similar to that for Proposition 2.1, except that the diffeomorphism $\Phi : U' \times W' \to U \times W$ in (2.1) only preserves $\mathbb{C}^n \times \mathcal{V}$. □

Then, for links we have a corresponding result provided the coefficient ring $R = \mathbf{k}$, a field of characteristic 0.

Proposition 2.4. *Suppose $f_i : \mathbb{C}^n, 0 \to \mathbb{C}^N, 0$, $i = 1, 2$ are $\mathcal{K}_{\mathcal{V}}$–equivalent. Let $\mathcal{V}_i = f_i^{-1}(\mathcal{V})$. Then, there is an isomorphism of graded vector spaces $\beta : H^*(L(\mathcal{V}_1); \mathbf{k}) \simeq H^*(L(\mathcal{V}_2); \mathbf{k})$ such that $\beta(\mathcal{B}_{\mathcal{V}}(f_1; \mathbf{k})) = \mathcal{B}_{\mathcal{V}}(f_2; \mathbf{k})$.*

Hence, $\mathcal{B}_{\mathcal{V}}(f_i; \mathbf{k})$ is, up to isomorphism, independent of the $\mathcal{K}_{\mathcal{V}}$–equivalence class of f_i.

Proof. The diffeomorphism $\Phi : U' \times W' \to U \times W$ induces diffeomorphisms graph$(f_i) \cap \mathcal{V}$ and graph$(f_i)\backslash\mathcal{V}$. These induce diffeomorphisms $U \cap \mathcal{V}_1 \simeq U' \cap \mathcal{V}_2$ and $U\backslash\mathcal{V}_1 \simeq U'\backslash\mathcal{V}_2$. These first induce isomorphisms $H^*(U'\backslash\mathcal{V}_2; \mathbf{k}) \simeq H^*(U\backslash\mathcal{V}_1; \mathbf{k})$. This continues to hold for sufficiently small balls using the argument in the proof of Proposition 2.1. As the homeomorphisms commute with f_i^*, we obtain the restriction isomorphism $\mathcal{C}_{\mathcal{V}}(f_1; \mathbf{k}) \simeq \mathcal{C}_{\mathcal{V}}(f_2; \mathbf{k})$.

Then, by choosing corresponding bases for these cohomology groups we obtain via the Kronecker pairings, isomorphisms with the homology groups of the complements. Then, associated to the isomorphisms between the $\mathcal{C}_{\mathcal{V}}(f_i; \mathbf{k})$, there is an induced isomorphism in reduced homology $\widetilde{\Gamma}_{\mathcal{V}}(f_1; \mathbf{k}) \simeq \widetilde{\Gamma}_{\mathcal{V}}(f_2; \mathbf{k})$. Lastly, Alexander duality induces isomorphisms of graded vector spaces $\mathcal{B}_{\mathcal{V}}(f_1; \mathbf{k}) \simeq \mathcal{B}_{\mathcal{V}}(f_2; \mathbf{k})$. □

3. Detecting the Nonvanishing of Characteristic Cohomology

We next ask for a singularity $\mathcal{V}_0$ of type $\mathcal{V}$, what will be the nonvanishing parts of the characteristic subalgebras $\mathcal{A}_{\mathcal{V}}(f_0; R)$, $\mathcal{C}_{\mathcal{V}}(f_0; R)$ and the characteristic cohomology $\mathcal{B}_{\mathcal{V}}(f_0; R)$? For the Milnor fiber, $\mathcal{A}_{\mathcal{V}}(f_0; R)$ is isomorphic to a quotient algebra of $H^*(F_w; R)$, but possibly it is just $H^0(\mathcal{V}_w; R)$. Similarly, for the complement $\mathcal{C}_{\mathcal{V}}(f_0; R)$, it is isomorphic to a quotient of $H^*(\mathbb{C}^N\backslash\mathcal{V}; R)$; and then we can determine a nonzero subgroup in $\mathcal{B}_{\mathcal{V}}(f_0; R)$ via Alexander duality.

We give a general method for detecting such non-zero subgroups of characteristic cohomology using "vanishing compact models" for both the Milnor fiber and complement.

Nonvanishing Characteristic Cohomology for the Milnor Fiber.

We consider a hypersurface singularity $\mathcal{V}, 0 \subset \mathbb{C}^N, 0$ with defining equation $H : \mathbb{C}^N, 0 \to \mathbb{C}, 0$ and Milnor fibration $H : H^{-1}(B^*_\eta) \cap B_{\delta_0} \to B^*_\eta$.

Definition 3.1. We say that $\mathcal{V}, 0$ has a *vanishing compact model for its Milnor fiber* if there is a compact space $Q_{\mathcal{V}}$, smooth curves $\gamma : [0, \eta) \to B_\eta$ satisfying $|\gamma(t)| = t$ and $\beta : [0, \eta) \to [0, \delta_0)$, monotonic with $\beta(0) = 0$, and an embedding into the Milnor fibration of H,

$\Phi : Q_{\mathcal{V}} \times (0, \delta) \hookrightarrow H^{-1}(B^*_\eta) \cap B_\delta$ such that:

i) each $H : H^{-1}(\overline{B}^*_{|\gamma(t)|}) \cap B_{\beta(t)} \to \overline{B}^*_{|\gamma(t)|}$ is again a Milnor fibration for H
ii) each $\Phi(Q_{\mathcal{V}} \times \{t\}) \subset F_w$ is a homotopy equivalence for F_w the Milnor fiber of i) over $w = \gamma(t)$.

This is the analogue of a basis of smoothly vanishing cycles for the isolated hypersurface case.

Next, with the situation as above, let $E \subseteq H^*(Q_{\mathcal{V}}; R)$ be a graded subgroup. We say that a compact subspace with inclusion map $\lambda_E : Q_E \subseteq Q_{\mathcal{V}}$ detects E in cohomology with R coefficients if the map on cohomology $\lambda_E^* : H^*(Q_{\mathcal{V}}; R) \to H^*(Q_E; R)$ induces an isomorphism from E to $H^*(Q_E; R)$. Then, we say that a *germ of an embedding* $i_E : \mathbb{C}^s, 0 \to \mathbb{C}^N, 0$ *detects* E if for sufficiently small $0 < \eta << \varepsilon < \delta$ there is a vanishing compact model $\Psi : Q_E \times (0, \delta) \hookrightarrow (H \circ i_E)^{-1}(B^*_\eta) \cap B_\varepsilon$ for the Milnor fibration of $H \circ i_E$ so that $i_E \circ \Psi = \Phi \circ (\lambda_E \times id)$, i.e. (3.1) commutes.

$$
\begin{array}{ccc}
Q_E \times (0, \delta) & \xrightarrow{\Psi} & (H \circ i_E)^{-1}(B^*_\eta) \cap B_\varepsilon \\
{\scriptstyle \lambda_E \times id} \Big\downarrow & & {\scriptstyle i_E} \Big\downarrow \\
Q_{\mathcal{V}} \times (0, \delta) & \xrightarrow{\Phi} & H^{-1}(B^*_\eta) \cap B_\delta
\end{array}
\tag{3.1}
$$

We then have the simple Lemma.

Lemma 3.2 (Detection Lemma for Milnor Fibers)**.** *Given* $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^N, 0$ *defining* $(\mathcal{V}_0, 0)$ *of type* $\mathcal{V}$*, suppose there is a germ* $g : \mathbb{C}^s, 0 \to \mathbb{C}^n, 0$ *such that* $f_0 \circ g$ *is* $\mathcal{K}_H$*-equivalent to a germ detecting* E*. Then,* $\mathcal{A}_{\mathcal{V}}(f_0, R)$ *contains a graded subgroup which is isomorphic to* E *via* $f_0^* : H^*(F_w; R) \to H^*(\mathcal{V}_w; R)$ *to* E*.*

Proof. We use the functoriality of $g^* : \mathcal{A}_{\mathcal{V}}(f_0, R) \to \mathcal{A}_{\mathcal{V}}(f_0 \circ g, R)$ given by Lemma 1.6. We do so using the representation by (1.6), with $\tilde{\varphi}$ representing g; and we first consider the case where the composition $f_0 \circ \tilde{\varphi}$ denotes $f_0 \circ g = i_E$. Provided $w = \gamma(t)$, with $0 < |w| < \eta$ is sufficiently small, there is the compact model $\Phi(Q_{\mathcal{V}} \times \{t\}) \subset F_w$. The composition gives as the embedding $i_E : (Q_E \times \{t\}) \subset (Q_{\mathcal{V}} \times \{t\}) \subset F_w$. In cohomology it maps $E \subseteq H^*(F_w; R)$ isomorphically to $H^*(Q_E \times \{t\}; R) \simeq H^*(Q_E; R)$.

Then,we compose the corresponding version of (1.6) with the map on cohomology factors through $(Q_E \times \{t\}) \subset S_w$ (for $\mathcal{S}_w$ the Milnor fiber of $H \circ i_E$). It will then send $E \subseteq H^*(F_w; R)$ isomorphically to the subgroup of the intermediate cohomology $H^*(\mathcal{V}_w; R)$. Thus, $\mathcal{A}_{\mathcal{V}}(f_0, R)$ contains this isomorphic copy of E via $\tilde{f}_{0,w}$.

Second, if instead $f_0 \circ g$ is $\mathcal{K}_H$-equivalent to i_E, by Proposition 2.1, there is an algebra isomorphism $\mathcal{A}_{\mathcal{V}}(i_E, R) \simeq \mathcal{A}_{\mathcal{V}}(f_0 \circ g, R)$. Then, $\mathcal{A}_{\mathcal{V}}(f_0 \circ g, R)$ contains a subspace isomorphic under an algebra isomorphism to E. Since this subspace is, up to an algebra isomorphism, the image of g^* of the image of $\tilde{f}^*_{0,w}(E)$ that image must be an isomorphic image of E. $\square$

Nonvanishing Characteristic Cohomology for the Complement and Link.

With the above notation, we consider the characteristic cohomology of the complement and link. We use the notation and neighborhoods given in the definition of the characteristic cohomology for the complement and link in §1 for $f_0(\overline{B}_{\varepsilon_0}) \subset B_{\delta_0}$. Then,

$$f_0^* : H^*(\overline{B}_{\delta_0} \backslash \mathcal{V}; R) \longrightarrow H^*(\overline{B}_\varepsilon \backslash \mathcal{V}_0; R)\,.$$

We introduce a corresponding vanishing compact model for the complement.

Definition 3.3. We say that $\mathcal{V}, 0$ has a *vanishing compact model for the complement* if there is a compact space $P_{\mathcal{V}}$, a smooth curve $\gamma : [0, \delta) \to [0, \delta_0)$, monotonic with $\gamma(0) = 0$, and an embedding into the complement of $\mathcal{V}$,
$\Phi : P_{\mathcal{V}} \times (0, \delta) \hookrightarrow B_{\delta_0} \backslash \mathcal{V}$ such that:

i) each $(\overline{B}_{\gamma(t)}, \overline{B}_{\gamma(t)} \cap \mathcal{V})$ again has a cone structure;
ii) $(\overline{B}_{\gamma(t')}, \overline{B}_{\gamma(t')} \cap \mathcal{V}) \subset (\overline{B}_{\gamma(t)}, \overline{B}_{\gamma(t)} \cap \mathcal{V})$ is a homotopy equivalence for $0 < t' < t$; and
iii) each $\Phi(P_{\mathcal{V}} \times \{t\}) \subset B_{\gamma(t)} \backslash \mathcal{V}$ is a homotopy equivalence.

Next, with the situation as above, let $E \subseteq H^*(P_{\mathcal{V}}; R)$ be a graded subgroup. We say that a compact subspace with inclusion map $\sigma_E : P_E \subseteq P_{\mathcal{V}}$ detects E in cohomology with R coefficients if the map on cohomology $\sigma_E^* : H^*(P_{\mathcal{V}}; R) \to H^*(P_E; R)$ induces an isomorphism from E to $H^*(P_E; R)$. Then, we say that a *germ of an embedding* $j_E : \mathbb{C}^s, 0 \to \mathbb{C}^N, 0$ *detects* E if for sufficiently small $0 < \varepsilon < \delta$ with $i_E(B_\varepsilon) \subset B_\delta$, there is a vanishing compact model $\Psi : P_E \times (0, \delta) \hookrightarrow B_\varepsilon \backslash j_E^{-1}(\mathcal{V})$ so that $j_E \circ \Psi = \Phi \circ (\sigma_E \times id)$. We then have the simple Lemma.

Lemma 3.4 (Second Detection Lemma)**.** *Given $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^N, 0$ defining $\mathcal{V}_0, 0$ of type $\mathcal{V}$, suppose there is a germ $g : \mathbb{C}^s, 0 \to \mathbb{C}^n, 0$ such that $f_0 \circ g$ is $\mathcal{K}_{\mathcal{V}}$-equivalent to a germ detecting E. Then, $\mathcal{C}_{\mathcal{V}}(f_0, R)$ contains a graded subgroup which is isomorphic to E via $f_0^* : H^*(B_\delta \backslash \mathcal{V}; R) \to H^*(B_\varepsilon \backslash \mathcal{V}_0); R)$.*

Proof. The proof is similar to that for Lemma 3.2 using instead the functoriality of $g^* : \mathcal{C}_{\mathcal{V}}(f_0, R) \to \mathcal{C}_{\mathcal{V}}(f_0 \circ g, R)$ given by Lemma 1.6. As we are really working with local cohomology, we must consider the cohomology groups of complements on varying neighborhoods of 0. By assumption there are vanishing compact models: $\Phi : P_{\mathcal{V}} \times (0, \delta) \hookrightarrow B_{\delta_0} \backslash \mathcal{V}$ and $\Psi : P_E \times (0, \delta) \hookrightarrow B_\varepsilon \backslash i_E^{-1}(\mathcal{V})$ so that $j_E \circ \Psi = \Phi \circ (\sigma_E \times id)$.

Provided $0 < \gamma(t) < \delta$ for δ sufficiently small, there is the compact model $\Phi(P_{\mathcal{V}} \times \{t\}) \subset B_{\gamma(t)} \backslash \mathcal{V}$. The composition $j_E : (P_E \times \{t\}) \subset (P_{\mathcal{V}} \times \{t\})$ is an embedding which in cohomology maps $E \subseteq H^*(P_{\mathcal{V}} \times \{t\}; R)$ isomorphically to $H^*(P_E \times \{t\}; R) \simeq H^*(P_E; R)$.

Then, we refer to corresponding version of (1.6) with $\tilde{\varphi}$ representing g and the composition $f_0 \circ \tilde{\varphi}$ denoting $f_0 \circ g = j_E$. We see that this composition further composed with the map on cohomology induced from $(P_E \times \{t\}) \subset B_\varepsilon \backslash i_E^{-1}(\mathcal{V})$ will then send $E \subseteq H^*(B_{\delta_0} \backslash \mathcal{V}; R)$ isomorphically to the graded subgroup of the intermediate cohomology $H^*(B_\varepsilon \backslash \mathcal{V}_0; R)$. Thus, $\mathcal{C}_{\mathcal{V}}(f_0, R)$ contains this isomorphic copy of E via f_0^*.

Also, if instead $f_0 \circ g$ is $\mathcal{K}_H$-equivalent to i_E, by Proposition 2.1, there is an algebra isomorphism $\mathcal{C}_{\mathcal{V}}(i_E, R) \simeq \mathcal{C}_{\mathcal{V}}(f_0 \circ g, R)$. Then, $\mathcal{C}_{\mathcal{V}}(f_0 \circ g, R)$ contains a subspace isomorphic under an algebra isomorphism to E. Since this subspace is, up to an algebra isomorphism, the image by g^* of the image of $f_0^*(E)$, that image must be the isomorphic image of E. □

Corollary 3.5. *Given $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^N, 0$ defining $\mathcal{V}_0, 0$ of type $\mathcal{V}$, suppose there is a germ $g : \mathbb{C}^s, 0 \to \mathbb{C}^n, 0$ such that $f_0 \circ g$ is $\mathcal{K}_{\mathcal{V}}$-equivalent to a germ detecting $E \subseteq \widetilde{H}^*(\mathbb{C}^N \backslash \mathcal{V}; \mathbf{k})$, for $\mathbf{k}$ a field of characteristic 0. Then, $\mathcal{B}_{\mathcal{V}}(f_0, \mathbf{k})$ contains a graded subgroup which is isomorphic via the Kronecker pairing and Alexander duality to the image of E via the isomorphisms $\widetilde{H}^j(B_\varepsilon \backslash \mathcal{V}_0; \mathbf{k}) \simeq \widetilde{H}^{2n-2-j}(S_\varepsilon^{2n-1} \cap \mathcal{V}_0; \mathbf{k})$.*

Proof. This is a consequence of the Second Detection Lemma 3.4 and the definition of $\mathcal{B}_{\mathcal{V}}(f_0, \mathbf{k})$ via the Kronecker pairing and Alexander duality. $\square$

4. Matrix Equivalence for the Three Types of Matrix Singularities

We will apply the results in previous sections to the cohomology for a matrix singularity $\mathcal{V}_0$ for any of the three types of matrices. We let M denote the space of $m \times m$ general matrices $M_n(\mathbb{C})$, resp. symmetric matrices $Sym_n(\mathbb{C})$, resp. skew-symmetric matrices $Sk_n(\mathbb{C})$. We also let $\mathcal{D}_m^{(*)}$ denote the variety of singular matrices for each case with $(*)$ denoting $()$ for general matrices, (sy) for symmetric matrices, or (sk) for skew-symmetric matrices. Also, the corresponding defining equations for the three cases are given by: det for the general and symmetric cases and the Pfaffian Pf for the skew-symmetric case. We generally denote the defining equation by $H : \mathbb{C}^N, 0 \to \mathbb{C}, 0$ for $\mathcal{V}$, where $M \simeq \mathbb{C}^N$ for appropriate N in each case and $\mathcal{V} = \mathcal{D}_m^{(*)}$.

Matrix Singularities Equivalences $\mathcal{K}_M$ and $\mathcal{K}_{HM}$.

There are several different equivalences that we shall consider for matrix singularities $f_0 : \mathbb{C}^n, 0 \to M, 0$ with $\mathcal{V}$ denoting the subvariety of singular matrices in M. The one used in classifications is $\mathcal{K}_M$*−equivalence:* We suppose that we are given an action of a group of matrices G on M. For symmetric or skew symmetric matrices, it is the action of $\mathrm{GL}_m(\mathbb{C})$ by $B \cdot A = B\,A\,B^T$. For general $m \times p$ matrices, it is the action of $\mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_p(\mathbb{C})$ by $(B, C) \cdot A = B\,A\,C^{-1}$. Given such an action, then the group $\mathcal{K}_M$ consists of pairs (φ, B), with φ a germ of a diffeomorphism of $\mathbb{C}^n, 0$ and B a holomorphic germ $\mathbb{C}^n, 0 \to G, I$. The action is given by

$$f_0(x) \mapsto f_1(x) \,=\, B(x) \cdot (f_0 \circ \varphi^{-1}(x))\,.$$

Although $\mathcal{K}_M$ is a subgroup of $\mathcal{K}_{\mathcal{V}}$, they have the same tangent spaces and their path connected components of their orbits agree (for example this is explained in [DP3, §2] because of the results due to Józefiak [J], Józefiak-Pragacz [JP], and Gulliksen-Negård[GN] as pointed out by Goryunov-Mond [GM]).

We next restrict to codimension 1 subgroups; let

$$GL_m(\mathbb{C})^{(2)} \;\stackrel{def}{=}\; \ker(\det \times \det : GL_m(\mathbb{C}) \times GL_m(\mathbb{C}) \to (\mathbb{C}^* \times \mathbb{C}^*)/\Delta\mathbb{C}^*)$$

where $\Delta\mathbb{C}^*$ is the diagonal subgroup. We then replace the groups for $\mathcal{K}_M$–equivalence by the subgroup $SL_m(\mathbb{C})$ for the symmetric and skew-symmetric case and for the general case the subgroup $GL_m(\mathbb{C})^{(2)}$. These restricted versions of equivalence preserve the defining equation H in each case. We denote the resulting equivalence groups by $\mathcal{K}_{HM}$, which are subgroups of the corresponding $\mathcal{K}_H$. As $\mathcal{K}_{HM}$ equivalences preserve H, they also preserve the Milnor fibers and the varieties of singular matrices $\mathcal{V}$. By the above referred to results, in each of the three cases, these $\mathcal{K}_{HM}$ also have the same tangent spaces as $\mathcal{K}_H$ in each case.

For each of the three cases of $m \times m$ matrices $M = M_m(\mathbb{C})$, resp. $Sym_m(\mathbb{C})$, resp. $Sk_m(\mathbb{C})$, the matrix singularities $\mathcal{V}_0$ defined by $f_0 : \mathbb{C}^n \to M, 0 = \mathbb{C}^N, 0$ have characteristic subalgebras that are defined by the results and discussion of §1. With $(*)$ denoting $()$ for general matrices, (sy) for symmetric matrices, or (sk) for skew-symmetric matrices, we will use

Abbreviated Notation for the Characteristic Cohomology for $\mathcal{V} = \mathcal{D}_m^{(*)}$*:*

$$\mathcal{A}^{(*)}(f_0; R) \; = \mathcal{A}_{\mathcal{D}_m^{(*)}}(f_0; R), \qquad \mathcal{C}^{(*)}(f_0; R) \; = \; \mathcal{C}_{\mathcal{D}_m^{(*)}}(f_0; R)\,,$$
$$\text{and} \quad \mathcal{B}^{(*)}(f_0; \mathbf{k}) \; = \; \mathcal{B}_{\mathcal{D}_m^{(*)}}(f_0; \mathbf{k})$$

for any coefficient ring R and field $\mathbf{k}$ of characteristic 0.

As a consequence of Proposition 2.1, since $\mathcal{K}_{HM}$ is a subgroup of $\mathcal{K}_H$, we have the following corollary.

Corollary 4.1. *For each of the three cases of the varieties of $m \times m$ singular matrices $\mathcal{V} = \mathcal{D}_m^{(*)}$, let $\mathcal{V}_0$ be defined by $f_0 : \mathbb{C}^n, 0 \to M, 0$, with M denoting the corresponding space of matrices. Then,*

a) *the characteristic subalgebra $\mathcal{A}^{(*)}(f_0; R)$ is, up to Milnor fiber cohomology isomorphism, an invariant of the $\mathcal{K}_{HM}$–equivalence class of f_0;*
b) *$\mathcal{B}^{(*)}(f_0; \mathbf{k})$ is, up an isomorphism of the cohomology of the link, an invariant of the $\mathcal{K}_M$–equivalence class of f_0; and*
c) *the characteristic subalgebra $\mathcal{C}^{(*)}(f_0; R)$ is, up to an isomorphism of the cohomology of the complement, an invariant of the $\mathcal{K}_M$–equivalence class of f_0.*

Hence, the structure of the cohomology of the Milnor fiber of $\mathcal{V}_0$ as a graded algebra (or graded module) over $\mathcal{A}^{(*)}(f_0; R)$ is, up to isomorphism, independent of the $\mathcal{K}_{HM}$–equivalence class of f_0

Before considering the cohomology of the Milnor fibers of the $\mathcal{D}_m(*)$, we first give an important property which implies that each of the $\mathcal{D}_m^{(*)}$ are H-holonomic in the sense of [D2], which gives a geometric condition that assists in proving that the matrix singularity is finitely $\mathcal{K}_{HM}$-determined (and hence finitely $\mathcal{K}_H$-determined). This will be a consequence of the fact that for all three cases the above groups act transitively on the strata of the canonical Whitney stratification of $\mathcal{D}_m^{(*)}$.

Lemma 4.2. *For each of the three cases of $m \times m$ general, symmetric and skew-symmetric matrices, the corresponding subgroups $GL_m(\mathbb{C})^{(2)}$, resp. $SL_m(\mathbb{C})$ act transitively on the strata of the canonical Whitney stratification of $\mathcal{D}_m^{(*)}$.*

Proof of Lemma 4.2. First, for the general case, let $A \in \mathcal{D}_m$ have rank $r < m$. We also denote the linear transformation on the space of column vectors defined by A to be denoted by L_A. Then, we let $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ denote a basis for $\mathbb{C}^m$ so that $\{\mathbf{v}_{r+1}, \dots, \mathbf{v}_m\}$ is a basis for $\ker(L_A)$. We also let $\{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ denote a basis for $\mathbb{C}^m$ so that $\mathbf{w}_j = L_A(\mathbf{v}_j)$ for $j = 1, \dots, r$. We let $b = \det(\mathbf{v}_1 \dots \mathbf{v}_m)$ and $c = \det(\mathbf{w}_1 \dots \mathbf{w}_m)$. Then, we let $B^{-1} = (\mathbf{v}_1, \dots, \mathbf{v}_{m-1}, \frac{c}{b}\mathbf{v}_m)$ and $C^{-1} = (\mathbf{w}_1 \dots \mathbf{w}_m)$. Then, $C \cdot A \cdot B^{-1} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, where I_r is the $r \times r$ identity matrix. Also,$\det(B) = \det(C) = c$ so $(B, C) \in GL_m(\mathbb{C})^{(2)}$. Thus, the each orbit of $GL_m(\mathbb{C}) \times GL_m(\mathbb{C})$, which consists of matrices of given fixed rank $< m$ is a stratum of the canonical Whitney stratification, is also an orbit of $GL_m(\mathbb{C})^{(2)}$.

For both the symmetric and skew-symmetric casesthe corresponding orbist under $GL_m(\mathbb{C})$ consist of matrices of given symmetric or skew-symmetric type of fixed rank $< m$; and they form stratam of the canonical Whitney stratification. We show that they are also orbits under the action of $SL_m(\mathbb{C})$.

For $A \in \mathcal{D}_m^{(sy)}$ of rank $r < m$, we consider the symmetric bilinear form $\psi(X, Y) = X^T \cdot A \cdot Y$ for column vectors in $\mathbb{C}^m$. We can find a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ for $\mathbb{C}^m$ so that $\psi(\mathbf{v}_i, \mathbf{v}_i) = 1$ for $i = 1, \dots, r$, $= 0$ for $i > r$, and $\psi(\mathbf{v}_i, \mathbf{v}_j) = 0$ if $i \neq j$. Then, let $b = \det(\mathbf{v}_1 \dots \mathbf{v}_m)$, we let $B^T = (\mathbf{v}_1, \dots, \mathbf{v}_{m-1}, \frac{1}{b}\mathbf{v}_m)$. Then $\det(B) = 1$ and $B \cdot A \cdot B^T = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$.

Lastly, for the skew-symmetric case the argument is similar, except for $A \in \mathcal{D}_m^{(sk)}$ of rank $r < m$, we consider the skew-symmetric bilinear form $\psi(X, Y) = X^T \cdot A \cdot Y$ for column vectors in $\mathbb{C}^m$ with even m and $r = 2k$. There is a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ for $\mathbb{C}^m$ so that $\psi(\mathbf{v}_{2i-1}, \mathbf{v}_{2i}) = 1$ for $i = 1, \dots, k$, and otherwise, $\psi(\mathbf{v}_i, \mathbf{v}_j) = 0$ for $i < j$. Then, let $b = \det(\mathbf{v}_1 \dots \mathbf{v}_m)$, we let $B^T = (\mathbf{v}_1, \dots, \mathbf{v}_{m-1}, \frac{1}{b}\mathbf{v}_m)$. Then $\det(B) = 1$ and $B \cdot A \cdot B^T = \begin{pmatrix} J_k & 0 \\ 0 & 0 \end{pmatrix}$, where J_k is the $r \times r$ block diagonal matrix with k 2×2-blocks of $J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. $\square$

5. Cohomology of the Milnor Fibers of the $\mathcal{D}_m^{(*)}$

We next recall results from [D3] and [D4] giving the cohomology structure of the Milnor fibers of the $\mathcal{D}_m^{(*)}$ for each of the three types of matrices. This includes: representing the Milnor fibers by global Milnor fibers, giving compact symmetric spaces as compact models for the homotopy types of the global Milnor fibers, giving the resulting cohomology for the symmetric spaces, geometrically representing the cohomology classes, and indicating the relation of the cohomology classes for different m.

Homotopy Type of Global Milnor fibers via Symmetric Spaces.
The global Milnor fibers for each of the three cases, which we denote by , F_m, resp. $F_m^{(sy)}$, resp. $F_m^{(sk)}$, are given by $H^{-1}(1)$ for $H : M, 0 \to \mathbb{C}, 0$ the defining equation for $\mathcal{D}_m^{(*)}$, which is det for the general of symmetric case and Pfaffian Pf for the skew-symmetric case. As shown in [D3] the Milnor fiber for the germ of H at 0 is diffeomorphic to the global Milnor fiber. The representation of the global Milnor fiber as a homogeneous space, by homotopy type as symmetric spaces, and compact models diffeomorphic to their Cartan models is given by [D4, Table 1], which we reproduce here.

Milnor Fiber $F_m^{(*)}$	Quotient Space	Symmetric Space	Compact Model $F_m^{(*)\,c}$	Cartan Model
F_m	$SL_m(\mathbb{C})$	SU_m	SU_m	F_m^c
$F_m^{(sy)}$	$SL_m(\mathbb{C})/SO_m(\mathbb{C})$	SU_m/SO_m	$SU_m \cap Sym_m(\mathbb{C})$	$F_m^{(sy)\,c}$
$F_m^{(sk)}, m = 2n$	$SL_{2n}(\mathbb{C})/Sp_n(\mathbb{C})$	SU_{2n}/Sp_n	$SU_m \cap Sk_m(\mathbb{C})$	$F_m^{(sk)\,c} \cdot J_n^{-1}$

Table 1. Global Milnor fiber, its representation as a homogenenous space, compact model as a symmetric space, compact model as subspace and Cartan model.

Tower Structures of Global Milnor fibers and Symmetric Spaces by Inclusion.

The global Milnor fibers for all cases, their symmetric spaces, and their compact models form towers via inclusions. These are given as follows. For the general and symmetric cases, there is the homomorphism $\tilde{j}_m : SL_m(\mathbb{C}) \hookrightarrow SL_{m+1}(\mathbb{C})$ sending $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. This can be identified with the inclusion of Milnor fibers $\tilde{j}_m : F_m \subset F_{m+1}$. Also, it restricts to give an inclusion $\tilde{j}_m : SU_m \hookrightarrow SU_{m+1}$ which are the compact models for the general case. Second, it induces an inclusion $\tilde{j}_m^{(sy)} : SL_m(\mathbb{C})/SO_m(\mathbb{C}) \hookrightarrow SL_{m+1}(\mathbb{C})/SO_{m+1}(\mathbb{C})$ which is an inclusion of Milnor fibers $\tilde{j}_m^{(sy)} : F_m^{(sy)} \hookrightarrow F_{m+1}^{(sy)}$. It also induces an inclusion of the compact homotopy models $\tilde{j}_m^{(sy)} : SU_m/SO_m(\mathbb{R}) \subset SU_{m+1}/SO_{m+1}(\mathbb{R})$ for the Milnor fibers.

For the skew symmetric case, the situation is slightly more subtle. First, the composition of two of the above successive inclusion homomorphisms for $SL_m(\mathbb{C})$ gives a homomorphism $SL_m(\mathbb{C}) \hookrightarrow SL_{m+2}(\mathbb{C})$ sending $A \mapsto \begin{pmatrix} A & 0 \\ 0 & I_2 \end{pmatrix}$ for the 2×2 identity matrix I_2. For even $m = 2k$, it induces an inclusion $\tilde{j}_m^{(sk)} : SL_m(\mathbb{C})/Sp_k(\mathbb{C}) \hookrightarrow SL_{m+2}(\mathbb{C})/Sp_{k+1}(\mathbb{C})$. However, the inclusion of Milnor fibers $\tilde{j}_m^{(sk)} : F_m^{(sk)} \hookrightarrow F_{m+2}^{(sk)}$ is given by the map sending $A \mapsto \begin{pmatrix} A & 0 \\ 0 & J_1 \end{pmatrix}$ for the 2×2 skew-symmetric matrix $J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. These two inclusions are related via the action of $SL_m(\mathbb{C})/Sp_k(\mathbb{C})$ on $F_m^{(sk)}$ which induces a diffeomorphism given by $B \mapsto B \cdot J_k \cdot B^T$ $(m = 2k)$. This also induces an inclusion of compact homotopy models $SU_m \cap Sk_m(\mathbb{C}) \subset SU_{m+2} \cap Sk_m(\mathbb{C})$. This inclusion commutes with both the inclusion of the Milnor fibers under the diffeomorphism given in [D3] by the action, and the inclusion of the Cartan models induced from the compact models after multiplying by J_k^{-1}, see Table 1. The Schubert decompositions for all three cases given in [D4] satisfy the additional property that they respect the inclusions.

Cohomology of Global Milnor fibers using Symmetric Spaces.

Next, we recall the form of the cohomology algebras for the global Milnor fibers. First, for the $m \times m$ matrices for the general case or skew-symmetric case (with $m = 2n$), with cohomology coefficients $R = \mathbb{Z}$, by Theorems [D4, Thm. 6.1] and [D4, Thm. 6.14],

$$H^*(F_m; \mathbb{Z}) \simeq \Lambda^* \mathbb{Z}\langle e_3, e_5, \ldots, e_{2m-1} \rangle \quad \text{general case} \tag{5.1}$$

$$H^*(F_m^{(sk)}; \mathbb{Z}) \simeq \Lambda^* \mathbb{Z}\langle e_5, e_9, \ldots, e_{4n-3} \rangle \quad \text{skew-symmetric case } (m = 2n) \tag{5.2}$$

and therefore these isomorphisms continue to hold with $\mathbb{Z}$ replaced by any coefficient ring R. Thus, for any coefficient ring R, $\mathcal{A}^{(*)}(f_0; R)$ is the quotient ring of a free exterior R-algebra on generators e_{2j-1}, for $j = 2, 3, \ldots, m$, resp. e_{4j-3} for $j = 2, 3, \ldots, n$.

For the $m \times m$ symmetric case there are two important cases where either $R = \mathbb{Z}/2\mathbb{Z}$ or is a field of characteristic zero. First, for the coefficient ring $R = \mathbf{k}$ a field of characteristic zero, the symmetric case breaks-up into two cases depending on whether m is even or odd (see [MT, Thm. 6.7 (2), Chap. 3] or Table 1 of [D3]).

$$H^*(F_m^{(sy)}; \mathbf{k}) \simeq \begin{cases} \Lambda^*\mathbf{k}\langle e_5, e_9, \dots, e_{2m-1}\rangle & \text{if } m = 2k+1 \\ \Lambda^*\mathbf{k}\langle e_5, e_9, \dots, e_{2m-3}\rangle\{1, e_m\} & \text{if } m = 2k \end{cases} \tag{5.3}$$

Here e_m is the Euler class of a m-dimensional real oriented vector bundle $\tilde{E}_m$ on the Milnor fiber $F_m^{(sy)}$. The vector bundle $\tilde{E}_m$ on the symmetric space $SU_m/SO_m(\mathbb{R})$ has the form $SU_m \times_{SO_m(\mathbb{R})} \mathbb{R}^m \to SU_m/SO_m(\mathbb{R})$ where the action of $SO_m(\mathbb{R})$ is given by the standard representation. This can be described as the *bundle of totally real subspaces* of $\mathbb{C}^m$, which is the bundle of m-dimensional real subspaces of $\mathbb{C}^m$ whose complexifications are $\mathbb{C}^m$.

In the second case for $R = \mathbb{Z}/2\mathbb{Z}$, by Theorem [D4, Thm. 6.15] using [MT, Thm. 6.7 (3), Chap. 3], we have

$$H^*(F_m^{(sy)}; \mathbb{Z}/2\mathbb{Z}) \simeq \Lambda^*\mathbb{Z}/2\mathbb{Z}\langle e_2, e_3, \dots, e_m\rangle \tag{5.4}$$

for generators $e_j = w_j(\tilde{E}_m)$, for $j = 2, 3, \dots, m$, for $w_j(\tilde{E}_m)$ the j-th Stiefel-Whitney class of the real oriented m-dimensional vector bundle $\tilde{E}_m$ above.

We summarize the structure of the characteristic subalgebra $\mathcal{A}^{(*)}(f_0; R)$ in each case with the following.

Theorem 5.1. *Let $f_0 : \mathbb{C}^n, 0 \to M, 0$ define a matrix singularity $\mathcal{V}_0, 0$ for M the space of $m \times m$ matrices which are either general, symmetric, or skew-symmetric (with $m = 2n$).*

i) *In the general and skew-symmetric cases, $\mathcal{A}^{(*)}(f_0; R)$ is a quotient of the free R-exterior algebra with generators given in* (5.1)
ii) *In the symmetric case with $R = \mathbb{Z}/2\mathbb{Z}$, $\mathcal{A}^{(sy)}(f_0; \mathbb{Z}/2\mathbb{Z})$ is the quotient of the free exterior algebra over $\mathbb{Z}/2\mathbb{Z}$ on generators $e_j = w_j(\tilde{E}_m)$, for $j = 2, 3, \dots, m$, for $w_j(\tilde{E}_m)$ the Stiefel-Whitney classes of the real oriented m-dimensional vector bundle $\tilde{E}_m$ on the Milnor fiber of $\mathcal{D}_m^{(sy)}$. Hence, $\mathcal{A}^{(*)}(f_0; \mathbb{Z}/2\mathbb{Z})$ is a subalgebra generated by the Stiefel-Whitney classes of the pull-back vector bundle $f_{0,w}^*(\tilde{E}_m)$ on $\mathcal{V}_w$.*
iii) *In the symmetric case with $R = \mathbf{k}$, a field of characteristic zero, $\mathcal{A}^{(sy)}(f_0; \mathbf{k})$ is a quotient of the $\mathbf{k}$-algebras in each of the cases in* (5.3).

Then, in each of these cases, the cohomology (with coefficients in a ring R) of the Milnor fiber of $\mathcal{V}_0$ has a graded module structure over the characteristic subalgebra $\mathcal{A}^{(*)}(f_0; R)$ of f_0.

Cohomology Relations Under Inclusions for Varying m.

We give the relations between the cohomology of the global Milnor fibers and the symmetric spaces for varying m under the induced inclusion mappings. The relations are the following.

Proposition 5.2. 1) *In the* general case, *for the inclusions $\tilde{j}_{m-1} : SU_{m-1} \hookrightarrow SU_m$ and $\tilde{j}_{m-1} : F_{m-1} \subset F_m$, $\tilde{j}_{m-1}^*$ is an isomorphism on the subalgebra generated by $\{e_{2i-1} : i = 2, \dots, m-1\}$ and $\tilde{j}_{m-1}^*(e_{2m-1}) = 0$.*

2) *In the* skew-symmetric case (with $m = 2n$), *for the inclusions $j_{m-2}^{(sk)}$: $SU_{2(n-1)}/Sp_{n-1} \hookrightarrow SU_{2n}/Sp_n$ and for Milnor fibers $\tilde{j}_{m-2}^{(sk)} : F_{m-2}^{(sk)} \hookrightarrow F_m^{(sk)}$, $\tilde{j}_{m-2}^{(sk)\,*}$ is an isomorphism on the subalgebra generated by $\{e_{4i-3} : i = 2, \dots, m-1\}$ and $\tilde{j}_{m-2}^{(sk)\,*}(e_{4m-3}) = 0$.*

3) *In the* symmetric case*, for the inclusion* $\tilde{\jmath}^{(sy)}_{m-1} : SU_{m-1}/SO_{m-1}(\mathbb{R}) \hookrightarrow SU_m/SO_m(\mathbb{R})$ *and for Milnor fibers* $\tilde{\jmath}^{(sy)}_{m-1} : F^{(sy)}_{m-1} \subset F^{(sy)}_m$*:*
   i) *for coefficients* $R = \mathbb{Z}/2\mathbb{Z}$*,* $\tilde{\jmath}^{(sy)\,*}_{m-1}$ *is an isomorphism on the subalgebra generated by* $\{e_i : i = 2, \dots, m-1\}$ *and* $\tilde{\jmath}^{(sy)\,*}_{m-1}(e_m) = 0$*;*
   iia) *for coefficients* $R = \mathbf{k}$*, a field of characteristic* 0*, if* $m = 2k$*, then* $\tilde{\jmath}^{(sy)\,*}_{m-1}$ *is an isomorphism on the subalgebra generated by* $\{e_{4i-3} : i = 2, \dots, k\}$*, and* $\tilde{\jmath}^{(sy)\,*}_{m-1}(e_m) = 0$*, and*
   iib) *if* $m = 2k+1$*, then* $\tilde{\jmath}^{(sy)\,*}_{m-1}$ *is an isomorphism on the subalgebra generated by* $\{e_{4i-3} : i = 2, \dots, k\}$*, and* $\tilde{\jmath}^{(sy)\,*}_{m-1}(e_{2m-1}) = 0$*,.*

Proof. For the general and skew-symmetric cases, the Schubert decomposition for the Cartan models $\mathcal{C}_m$ and $\mathcal{C}^{(sk)}_m$ for successive m given in [D4] preserves the inclusions and the homology properties. In these two cases the result follows from the resulting identified Kronecker dual cohomology classes [D4, §6].

For the symmetric case and for $\mathbb{Z}/2\mathbb{Z}$-coefficients, an analogous Schubert decomposition gives the corresponding result. The remaining symmetric case for coefficients $\mathbf{k}$ a field of characteristic 0 does not follow in [D4] from the Schubert decomposition. Instead, the computation of the cohomology of the symmetric space given in [MT, Chap. 3] yields the result. In fact the algebraic computations in [MT, Chap. 3] also give the results for the other cases. □

Vanishing Compact Models for the Milnor Fibers of $\mathcal{D}^{(*)}_m$.

We can use the preceding compact models for the Milnor fibers and complements to give vanishing compact models for both cases and for detecting nonvanishing subalgebras of the characteristic subalgebras. From the above, let $F^{(*),c}_M$ denote the compact models for the individual global Milnor fibers $F^{(*)}_M$. We define $\Phi : F^{(*),c}_m \times (0,1] \to H^{-1}((0,1])$ sending $\Phi(A,t) = t \cdot A$. Also, let $E = \Lambda^* R\{e_{i_1}, \dots, e_{i_\ell}\}$ denote the exterior subalgebra of $H^*(F^{(*),c}_m; R)$ on generators of the ℓ lowest degrees. We also let $\lambda_E : F^{(*),c}_\ell \to F^{(*),c}_m$ denote the compositions $\tilde{\jmath}^{(*)}_{m-1} \circ \cdots \circ \tilde{\jmath}^{(*)}_\ell$. Then, by Proposition 5.2, λ^*_E induces an isomorphism from E to its image. Our goal is to first show that an appropriate restriction of Φ to a subinterval $(0, \delta)$ will provide a vanishing compact model for $F^{(*)}_M$; and moreover, we will use λ^*_E to give a germ which detects E. First, we give vanishing compact models for each case as follows.

Proposition 5.3. *A vanishing compact model for the Milnor fiber for* $\mathcal{D}^{(*)}_M$ *is given for sufficiently small* $0 < \delta << \varepsilon$ *by* $\Phi : F^{(*),c}_m \times (0, \varepsilon] \to H^{-1}((0, \varepsilon])$ *sending* $\Phi(A,t) = t \cdot A$*.*

Proof. We begin by first making a few observations about the global Milnor fibers. For M one of the spaces of $m \times m$ matrices, we consider $H : M, 0 \to \mathbb{C}, 0$ the defining equation for $\mathcal{D}^{(*)}_m$ ($H = \det$ or Pf). Then, the global Milnor fiber is $F^{(*)}_m = H^{-1}(1)$. Now we can consider multiplication in M by a constant $a \neq 0$. As H is homogeneous, if $A \in F^{(*)}_m$, then $a \cdot A \in H^{-1}(a^m)$ in the general or symmetric cases, or in the skew-symmetric cases $H^{-1}(a^k)$ where $m = 2k$.

We also observe that multiplication by a is a diffeomorphism between these two Milnor fibers. We denote the image of $F^{(*)}_m$ by multiplication by a by $aF^{(*)}_m$. Then, by e.g. the proof of [D3, Lemma 1.2], given $\delta > 0$, there is an $a > 0$ so that $aF_m^{(*)} \cap B_\delta$ is the local Milnor fiber of $\mathcal{V}_0$, $aF_m^{(*)}$ is transverse to the spheres of radii $\geq \delta$, and $aF_m^{(*)} \cap B_\delta \subset aF_m^{(*)}$ is a homotopy equivalence.

Also, we have the compact homotopy models which occur as submanifolds of SU_m of the form SU_m for the general case, resp. $SU_m \cap Sym_m(\mathbb{C})$ for the symmetric case, resp. $SU_m \cap Sk_m(\mathbb{C})$ for the skew-symmetric case. Now, for the standard Euclidean norm on $M_n(\mathbb{C})$, $\|A\| = \sqrt{m}$ for $A \in SU_m$. Then, as well this holds for $SU_m \cap Sym_m(\mathbb{C})$, and for $SU_m \cap Sk_m(\mathbb{C})$. We denote the compact model in $F_m^{(*)}$ by $F_m^{(*)\, c}$. Then, in each case if $M \simeq \mathbb{C}^N$, $F_m^{(*)\, c} \subset S^{2N-1}_{\sqrt{m}}$, the sphere of radius $\sqrt{m}$. Thus, $aF_m^{(*)\, c} \subset S^{2N-1}_{a\sqrt{m}}$.

Then, we first choose $0 < \eta << \delta < 1$ so that $H : H^{-1}(B_\eta^*) \cap B_\delta \to B_\eta^*$ is the Milnor fibration of H.

We choose $0 < a < \eta$ so that also $a\sqrt{m} < \delta$. Then, we observe the composition $aF_m^{(*)\, c} \subset aF_m^{(*)} \cap B_\delta \subset aF_m^{(*)}$ is a homotopy equivalence. Hence, The restriction $\Phi : F_m^{(*)\, c} \times (0, a) \to H^{-1}(B_a^*) \cap B_\delta \to B_a^*$ restricts to a homotopy equivalence for each $0 < t < a$ and so gives a vanishing compact model. □

In light of Theorem 5.1, there are several natural problems to be solved involving the characteristic cohomology for matrix singularities of each of the types.

Problems for the Characteristic Cohomology of the Milnor Fibers of Matrix Singularities:

1) Determine the characteristic subalgebras as the images of the exterior algebras by detecting which monomials map to nonzero elements in $H^*(\mathcal{V}_w; R)$.
2) Identify geometrically these non-zero monomials in 1) via the pull-backs of the Schubert classes.
3) For the symmetric case with $\mathbb{Z}/2\mathbb{Z}$-coefficients, compute the Stiefel-Whitney classes of the pull-back of the vector bundle $\tilde{E}_m$.
4) Determine a set of module generators for the cohomology of the Milnor fibers as modules over the characteristic subalgebras.

We will give partial answers to these problems in the next sections.

6. Kite Spaces of Matrices for Given Flag Structures

We begin by introducing for a flag of subspaces for $\mathbb{C}^m$, a *linear kite subspace* of size k in the space of $m \times m$ matrices of any of the three types: general $M_m(\mathbb{C})$, symmetric $Sym_m(\mathbb{C})$, or skew-symmetric $Sk_m(\mathbb{C})$ (with m even). We initially consider the standard flag for $\mathbb{C}^m$, given by $0 \subset \mathbb{C} \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^{m-1} \subset \mathbb{C}^m$. We choose coordinates $\{x_1, \cdots, x_m\}$ for $\mathbb{C}^m$ so that $\{x_1, \cdots, x_k\}$ are coordinates for $\mathbb{C}^k$ for each k.

We let $E_{i,j}$ denote the $m \times m$ matrix with entry 1 in the (i, j)-position and 0 otherwise. We also let $E^{(sy)}_{i,j} = E_{i,j} + E_{j,i}$, $i < j$, or $E^{(sy)}_{i,i} = E_{i,i}$ for the space of symmetric matrices; and $E^{(sk)}_{i,j} = E_{i,j} - E_{j,i}$, for $i < j$. Then, we define

Definition 6.1. For each of the three types of $m \times m$ matrices and the standard flag of subspaces of $\mathbb{C}^m$, the corresponding *linear kite subspace of size ℓ* is the linear subspace of the space of matrices defined as follows:

i) For $M_m(\mathbb{C})$, it is the linear subspace $\mathbf{K}_m(\ell)$ spanned by

$$\begin{pmatrix} * & \cdots & * & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & 0 & \cdots & 0 \\ * & \cdots & * & 0 & \cdots & 0 \\ 0 & \cdots & 0 & * & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & * \end{pmatrix}$$

FIGURE 1. Illustrating the form of elements of a linear kite space of size ℓ in either the space of general matrices or symmetric matrices. For general matrices the upper left matrix of size $\ell \times \ell$ is a general matrix, while for symmetric matrices it is symmetric.

$$\{E_{i,j} : 1 \leq i, j \leq \ell\} \cup \{E_{i,i} : \ell < i \leq m\}$$

ii) For $Sym_m(\mathbb{C})$, it is the linear subspace $\mathbf{K}_m^{(sy)}(\ell)$ spanned by

$$\{E_{i,j}^{(sy)} : 1 \leq i \leq j \leq \ell\} \cup \{E_{i,i} : \ell < i \leq m\}$$

iii) For $Sk_m(\mathbb{C})$ with m even, for ℓ also even, it is the linear subspace $\mathbf{K}_m^{(sk)}(\ell)$ spanned by

$$\{E_{i,j}^{(sk)} : 1 \leq i < j \leq \ell\} \cup \{E_{2i,2i+1}^{(sk)} : \ell < 2i < m\}$$

Furthermore, we refer to the germ of the inclusion $i_m^{(*)}(\ell) : \mathbf{K}_m^{(*)}(\ell), 0 \to M, 0$, for each of the three cases as a *linear kite map of size ℓ*.

The general form of elements "the kites" in the linear kite subspaces have the form given in (6.1).

$$Q_{\ell,m-\ell} \;=\; \begin{pmatrix} A_\ell & 0_{\ell,m-\ell} \\ 0_{m-\ell,\ell} & D_{m-\ell} \end{pmatrix} \tag{6.1}$$

where A_ℓ is an $\ell \times \ell$-matrix which denotes an arbitrary matrix in either $M_\ell(\mathbb{C})$, resp. $Sym_\ell(\mathbb{C})$, resp. $Sk_\ell(\mathbb{C})$; and $0_{r,s}$ denotes the zero $r \times s$ matrix.. Also, $D_{m-\ell}$ denotes an arbitrary $(m-\ell) \times (m-\ell)$ diagonal matrix in the general or symmetric case as in Figure 6. In the skew symmetric case, $D_{m-\ell}$ denotes the 2×2 block diagonal matrix with skew-symmetric blocks of the form given by (6) as in Figure 2.

$$J_1(*) \;=\; \begin{pmatrix} 0 & * \\ -* & 0 \end{pmatrix}$$

with "* "denoting an arbitrary entry.

We next extend this to general flags, and then to nonlinear subspaces as follows. For each of the three types of matrices $M =$, resp. $M_m(\mathbb{C})$, resp. $Sym_m(\mathbb{C})$, resp. $Sk_m(\mathbb{C})$ (with m even).

$$\begin{pmatrix} * & \cdots & * & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & 0 & \cdots & 0 \\ * & \cdots & * & 0 & \cdots & 0 \\ 0 & \cdots & 0 & J_1(*) & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & J_1(*) \end{pmatrix}$$

FIGURE 2. Illustrating the form of elements of a linear "skew-symmetric kite"space of size ℓ (with ℓ even) in the space of skew-symmetric matrices. The upper left $\ell \times \ell$ matrix is a skew-symmetric matrix.

$$\begin{pmatrix} x_{1,1} + 2x_{4,4}x_{1,3} & x_{1,2} + 2x_{4,4}x_{2,3} & x_{1,3} + 2x_{4,4}x_{3,3} & 0 \\ x_{1,2} + 2x_{4,4}x_{2,3} & x_{2,2} - 35x_{1,2}x_{4,4} & x_{2,3} & (7x_{1,2} - 5)x_{4,4} \\ x_{1,3} + 2x_{4,4}x_{3,3} & x_{2,3} & x_{3,3} & 0 \\ 0 & (7x_{1,2} - 5)x_{4,4} & 0 & x_{4,4} \end{pmatrix}$$

FIGURE 3. An example of an unfurled kite map of size 3 into 4×4 symmetric matrices.

Definition 6.2. An *unfurled kite map* of given matrix type is any element of the orbit of $i_m^{(*)}(\ell)$, for $(*) = ()$, resp. (sy), resp.(sk), under the corresponding equivalence group $\mathcal{K}_{HM}$.

A germ $f_0 : \mathbb{C}^n, 0 \to M, 0$ *contains a kite map of size* ℓ for each of the three cases if there is a germ of an embedding $g : \mathbf{K}_m^{(*)}(\ell), 0 \to \mathbb{C}^n, 0$ such that $f_0 \circ g$ is an unfurled kite map.

Remark 6.3. We note that unfurled kite maps have the property that the standard flag can be replaced by a general flag; and moreover, the flag and linear kite space can undergo nonlinear deformations. These can be performed by iteratively applying appropriate row and column operations using elements of the local ring of germs on $\mathbb{C}^n, 0$ instead of constants.

A simple example of an unfurled kite map is given in Figure 3.

7. Detecting Characteristic Cohomology using Kite Spaces of Matrices

In §5, we gave in equations (5.1), (5.3), and (5.4) the cohomology of the Milnor fibers for the $\mathcal{D}_m^{(*)}$ for each of the three types of matrices. Thus, as for any matrix singularity $f_0 : \mathbb{C}^n, 0 \to M, 0$, by Theorem 5.1 the characteristic subalgebra is a quotient of the corresponding algebra. As previously in §5, we let $E = \Lambda^* R\{e_{i_1}, \dots, e_{i_\ell}\} \subseteq H^*(F_m^{(*),c}; R)$ denote an exterior algebra on generators of the ℓ lowest degrees. Then, using the map λ_E given before Proposition 5.3, λ_E^* induces an isomorphism from E to its image. We next use λ_E to show that for germs f_0 containing a kite map of size ℓ for each case detects E in $\mathcal{A}^{(*)}(f_0, R)$.

Theorem 7.1. *Let $f_0 : \mathbb{C}^n, 0 \to M, 0$ define an $m \times m$ matrix singularity of one of the three types.*

a) *In the case of general matrices, if f_0 contains an unfurled kite map of size $\ell < m$, then $\mathcal{A}(f_0, R)$ contains an exterior algebra of the form*
$$\Lambda^* R\langle e_3, e_5, \dots, e_{2\ell-1}\rangle \, .$$
on $\ell - 1$ generators.

b) *In the case of skew-symmetric matrices (with m even), if f_0 contains an unfurled skew-symmetric kite map of size $\ell (= 2k) < m$, then $\mathcal{A}^{(sk)}(f_0, R)$ contains an exterior algebra of the form*
$$\Lambda^* R\langle e_5, e_9, \dots, e_{4k-3}\rangle \, .$$
on $k - 1$ generators.

c) *In the case of symmetric matrices, if f_0 contains an unfurled symmetric kite map of size $\ell < m$, then $\mathcal{A}^{(sy)}(f_0, R)$ contains an exterior algebra of one of the forms*

$\Lambda^* \mathbf{k}\langle e_3, e_5, \dots, e_{2\ell-1}\rangle$ *if $R = \mathbf{k}$ is a field of characteristic 0,*

$\Lambda^* \mathbb{Z}/2\mathbb{Z}\langle e_2, e_3, \dots, e_\ell\rangle$ *if $R = \mathbb{Z}/2\mathbb{Z}$,*

Remark 7.2. In the symmetric case, it follows from c) that if f_0 contains an unfurled symmetric kite map of size $\ell < m$, then the Stiefel-Whitney classes of the pull-back bundle $w_i(f_{0,w}^*(\tilde{E}_m))$ on $\mathcal{V}_w$ are non-vanishing for $i = 2, \dots, \ell$.

Proof. By Theorem 5.1 and the Detection Lemma 3.2, it is sufficient to show that the corresponding kite maps of each type detect the corresponding exterior subalgebra. We use the notation from the proof of Proposition 5.3 which gave the vanishing compact models for the Milnor fibers in each case.

Then, we choose $0 < \eta << \varepsilon < \delta < 1$ so that $H : H^{-1}(B_\eta^*) \cap B_\delta \to B_\eta^*$ is the Milnor fibration of H and $H \circ i_m^{(*)}(\ell) : (H \circ i_m^{(*)}(\ell))^{-1}(B_\eta^*) \cap B_\varepsilon \to B_\eta^*$ is the Milnor fibration of $H \circ i_m^{(*)}(\ell)$. We also choose $0 < a < \eta$ so that $a\sqrt{m} < \varepsilon$.

Then there are the following inclusions.

$$aF_\ell^{(*)\, c} \;\subset\; i_m^{(*)}(\ell)(H \circ i_m^{(*)}(\ell))^{-1}(a^r) \cap B_\varepsilon) \;\subset\; aF_m^{(*)} \cap B_\varepsilon \;\subset\; aF_m^{(*)} \, , \tag{7.1}$$

where $r = m$ in the general or symmetric case or $r = \frac{m}{2}$ in the skew-symmetric case. The composition of inclusions $F_\ell^{(*)\, c} \subset F_m^{(*)\, c} \subset F_m^{(*)}$ commutes with multiplication by a as in Figure 10.1 where each vertical map is a diffeomorphism given by multiplication by a.

$$\begin{tikzcd} F_\ell^{(*)\, c} \arrow[r] \arrow[d] & F_m^{(*)\, c} \arrow[r] \arrow[d] & F_m^{(*)} \arrow[d] \\ aF_\ell^{(*)\, c} \arrow[r, "i_m^{(*)}(\ell)"] & aF_m^{(*)\, c} \arrow[r] & aF_m^{(*)} \end{tikzcd} \tag{7.2}$$

Also, $i_m^{(*)}(\ell)$ in the bottom row is given by the map in (7.3).

$$aA \;\mapsto\; aQ_{\ell, m-\ell} \;=\; \begin{pmatrix} aA_\ell & 0_{\ell, m-\ell} \\ 0_{m-\ell, \ell} & aD_{m-\ell} \end{pmatrix} \tag{7.3}$$

Then, by Proposition 5.2 the induced homomorphisms in cohomology for the top row of (10.1) restrict to an isomorphism on the corresponding exterior subalgebra of $H^*(F_m^{(*)}; R)$ onto the cohomology $H^*(F_\ell^{(*)\, c}; R)$, and vanishing on the remaining generators. Hence, as the vertical diffeomorphisms induce isomorphisms on cohomology, the induced homomorphisms on cohomology for the bottom row have the same property. Lastly, in (7.1), the induced homomorphisms in cohomology restrict to an isomorphism on the corresponding exterior subalgebra of $H^*(F_m^{(*)\, c}; R)$ to $H^*(aF_\ell^{(*)\, c}; R)$. Thus the induced homomorphism to the Milnor fiber of $i_m^{(*)}(\ell)$,

$$H^*(aF_m^{(*)\, c}; R) \longrightarrow H^*(H \circ i_m^{(*)}(\ell))^{-1}(a^r) \cap B_\varepsilon; R)$$

restricts to an isomorphism of the corresponding exterior algebra onto its image. Thus, the cohomology of the Milnor fiber of $H \circ i_m^{(*)}(\ell)$ contains the claimed exterior subalgebra. Thus, the flag map $i_m^{(*)}(\ell)$ detects the corresponding exterior algebra, so the result follows by the Detection Lemma. $\square$

8. Examples of Matrix Singularities Exhibiting Characteristic Cohomology

We consider several examples illustrating Theorem 7.1.

$$\begin{pmatrix} x_{1,1} + 2x_{4,4}x_{1,3} & x_{1,2} + 2x_{4,4}x_{2,3} & x_{1,3} + 2x_{4,4}x_{3,3} & y_1 \\ x_{1,2} + 2x_{4,4}x_{2,3} & x_{2,2} - 35x_{1,2}x_{4,4} & x_{2,3} + y_1 x_{1,1}^2 & (7x_{1,2} - 5)x_{4,4} \\ x_{1,3} + 2x_{4,4}x_{3,3} & x_{2,3} + y_1 x_{1,1}^2 & x_{3,3} + y_2 x_{2,2}^2 & y_2 \\ y_1 & (7x_{1,2} - 5)x_{4,4} & y_2 & x_{4,4} \end{pmatrix}$$

Figure 4. An example of a germ f_0 containing an unfurled kite map of size 3 into 4×4 symmetric matrices in Figure 3.

Example 8.1. Let $f_0; \mathbb{C}^9, 0 \to Sym_4(\mathbb{C}), 0$ be defined by $f_0(\mathbf{x}, \mathbf{y})$ given by the matrix in Figure 4 for $\mathbf{x} = (x_{1,1}, x_{1,2}, x_{1,3}, x_{2,2}, x_{2,3}, x_{3,3}, x_{4,4})$ and $\mathbf{y} = (y_1, y_2)$. We let $\mathcal{V}_0 = f_0^{-1}(\mathcal{D}_4^{(sy)})$. This is given by the determinant of the matrix in Figure 4 defining f_0. Then, $\mathcal{V}_0$ has singularities in codimension 2. We observe that when $\mathbf{y} = (0, 0)$ we obtain the unfurled kite map in Figure 3. Thus, by Theorem 7.1, the Milnor fiber of $\mathcal{V}_0$ has cohomology with $\mathbb{Z}/2\mathbb{Z}$ coefficients containing the subalgebra $\Lambda^*\mathbb{Z}/2\mathbb{Z}\langle e_2, e_3\rangle$, so there is $\mathbb{Z}/2\mathbb{Z}$ cohomology in degrees 2, 3, and 5. We also note that $e_j = w_j(f_{0,w}^* \tilde{E}_4)$ so that one consequence is that the second and third Stiefel-Whitney classes of the pullback of the vector bundle $\tilde{E}_4$ are non-zero.

For coefficients a field $\mathbf{k}$ of characteristic 0, the cohomology of the Milnor fiber of $\mathcal{V}_0$ has an exterior algebra $\Lambda^*\mathbf{k}\langle e_5\rangle$, so there is a $\mathbf{k}$ generator e_5 in degree 5.

By Kato-Matsumota [KM], as singularities have codimension 2, the Milnor fiber is simply connected. Then, we can use the preceding to deduce information about the integral cohomology of the Milnor fiber from the universal coefficient theorem. It must have rank at least 1 in dimension 5, and it has 2-torsion in dimension 2.

Second, we consider a general matrix singularity.

Example 8.2. We let $f_0; \mathbb{C}^{21}, 0 \to M_5(\mathbb{C}), 0$ be defined with $f_0(\mathbf{x}, \mathbf{y})$ given by the matrix in Figure 5 for $\mathbf{x} = (x_{1,1}, \dots, x_{4,4}, x_{5,5})$ and $\mathbf{y} = (y_1, y_2, y_3, y_4)$. In this example we require that $g_i(\mathbf{x}, 0) \equiv 0$ for each i . We let $\mathcal{V}_0 = f_0^{-1}(\mathcal{D}_5)$. This is given by the determinant of the matrix in Figure 5 defining f_0. Then, the $\mathcal{V}_0$ has singularities in codimension 4 in $\mathbb{C}^{21}$; hence by Kato-Matsumoto, the Milnor fiber

$$\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & g_1(\mathbf{x},\mathbf{y}) \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & g_2(\mathbf{x},\mathbf{y}) \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & g_3(\mathbf{x},\mathbf{y}) \\ x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} & g_4(\mathbf{x},\mathbf{y}) \\ y_1 & y_2 & y_3 & y_4 & x_{5,5} \end{pmatrix}$$

FIGURE 5. An example of a germ f_0 in Example 8.2, containing a linear kite map of size 4 into 5×5 general matrices with $g_i(\mathbf{x}, 0) \equiv 0$ for each i).

is 2-connected. We observe that when $\mathbf{y} = (0,0,0,0)$ we obtain the linear kite map $i_5(4)$. Thus, by Theorem 7.1, the Milnor fiber of $\mathcal{V}_0$ has characteristic cohomology with integer coefficients containing the subalgebra $\Lambda^*\mathbb{Z}\langle e_3, e_5, e_7\rangle$. Hence, the integer cohomology has rank at least 1 in dimensions $0, 3, 5, 7, 8, 10, 12, 15$. We cannot determine at this point whether the generator e_9 maps to a nonzero element in the cohomology of the Milnor fiber of $\mathcal{V}_0$. Even if it does, there are several products involving e_9 in exterior algebra for the cohomology of $\mathcal{D}_5$ must map to 0, as the Milnor fiber is homotopy equivalent to a CW-complex of dimension 20.

9. Characteristic Cohomology for the Complements and Links of Matrix Singularities

We now turn to the characteristic cohomology of the complement and link for matrix singularities of all three types. In order to apply the earlier results to the cases of matrix singularities, we first recall in Table 2 the cohomology, with coefficients a field $\mathbf{k}$ of characteristic 0, of the complements and links as given in [D3, table 2]. We will then use the presence of kite maps to detect both subalgbras of $\mathcal{C}^{(*)}(f_0, R)$ for the complements and subgroups of $\mathcal{B}^{(*)}(f_0, \mathbf{k})$ for the links.

Theorem 9.1. *Let $f_0 : \mathbb{C}^n, 0 \to M, 0$ define a matrix singularity $\mathcal{V}_0$ of any of the three types. If f_0 contains a kite map of size ℓ, then the characteristic cohomology of the complement $\mathcal{C}^{(*)}(f_0, \mathbf{k})$, for a field $\mathbf{k}$ of characteristic 0, contains an exterior algebra given by Table 3.*

Furthermore, the characteristic cohomology of the link $\mathcal{B}^{(*)}(f_0, \mathbf{k})$, as a graded vector space contains the graded subspace given by truncating the exterior subalgebra of $\mathcal{C}^{(*)}(f_0, \mathbf{k})$ listed in column 2 of Table 3 in the top degree and shifting by the amount listed in the last column.

For the complements in the general and skew-symmetric cases, $\mathbf{k}$ may be replaced by any coefficient ring R.

Remark 9.2. In what follows to simplify statements, instead of referring to the complement of $\mathcal{V}_0, 0 \subset \mathbb{C}^n, 0$ as $B_\varepsilon \backslash \mathcal{V}_0$ for sufficiently small $\varepsilon > 0$, we will just refer to the complement as $\mathbb{C}^n \backslash \mathcal{V}_0$, with the understanding that it is restricted to a sufficiently small ball.

Proof of Theorem 9.1. The proof is similar to that for Theorem 7.1. As the statements are independent of f_0 in a given $\mathcal{K}_{\mathcal{V}}$-equivalence class, we may apply an element of $\mathcal{K}_H$ to obtain an f_0 containing a linear kite map. It is sufficient to show, as for the case of Milnor fibers, that the linear kite map detects the indicated

Determinantal Hypersurface	Complement $M \backslash \mathcal{D}$	$H^*(M\backslash\mathcal{D}, \mathbf{k}) \simeq H^*(K/L, \mathbf{k})$	Shift
$\mathcal{D}_m^{sy}$ (m = 2k+1)	$GL_m(\mathbb{C})/O_m(\mathbb{C}) \sim U_m/O_m(\mathbb{R})$	$\Lambda^*\mathbf{k}\langle e_1, e_5, \dots, e_{2m-1}\rangle$	$\binom{m+1}{2} - 2$
$\mathcal{D}_m^{sy}$ (m = 2k)	$GL_m(\mathbb{C})/O_m(\mathbb{C})$	$\Lambda^*\mathbf{k}\langle e_1, e_5, \dots, e_{2m-3}\rangle$	$\binom{m+1}{2} + m - 2$
$\mathcal{D}_m$	$GL_m(\mathbb{C}) \sim U_m$	$\Lambda^*\mathbf{k}\langle e_1, e_3, \dots, e_{2m-1}\rangle$	$m^2 - 2$
$\mathcal{D}_m^{sk}$ (m = 2k)	$GL_{2k}(\mathbb{C})/Sp_k(\mathbb{C}) \sim U_{2k}/Sp_k$	$\Lambda^*\mathbf{k}\langle e_1, e_5, \dots, e_{2m-3}\rangle$	$\binom{m}{2} - 2$

TABLE 2. The cohomology of the complements $M\backslash\mathcal{D}$ and links $L(\mathcal{D})$ for each determinantal hypersurface $\mathcal{D}$. The complements, are homotopy equivalent to the quotients of maximal compact subgroups K/L with cohomology given in the third column, where the generators of the cohomology e_k are in degree k; and the structure is an exterior algebra. For the links $L(\mathcal{D})$, the cohomology is isomorphic as a vector space to the cohomology of the complement truncated in the top degree and shifted by the degree indicated in the last column.

subalgebra in $\mathcal{C}^{(*)}(f_0, \mathbf{k})$, and then apply Alexander duality for the result for the link.

By the results in [D3] summarized in Table 2, the complement $M\backslash\mathcal{D}_m^{(*)}$ is given by a homogeneous space G/H which has as a compact homotopy model (K/L) where $K = U_m$ for each of the cases. For successive values of m, we have for the three cases the successive inclusions:

i) for the general case, $GL_m(\mathbb{C}) \hookrightarrow GL_{m+1}(\mathbb{C})$ by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$;

ii) for the symmetric case, $GL_m(\mathbb{C}) \hookrightarrow GL_{m+1}(\mathbb{C})$ sending $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ induces an inclusion $GL_m(\mathbb{C})/O_m(\mathbb{C}) \hookrightarrow GL_{m+1}(\mathbb{C})/O_{m+1}(\mathbb{C})$;

iii) for even $m = 2k$, $GL_m(\mathbb{C}) \hookrightarrow GL_{m+2}(\mathbb{C})$ sending $A \mapsto \begin{pmatrix} A & 0 \\ 0 & I_2 \end{pmatrix}$, for I_2 the 2×2 identity matrix, induces an inclusion $GL_m(\mathbb{C})/Sp_k(\mathbb{C}) \hookrightarrow GL_{m+2}(\mathbb{C})/Sp_{k+1}(\mathbb{C})$.

Then, these are obtained by the action of $GL_m(\mathbb{C})$ on the appropriate spaces of matrices. They restrict to the compact homogenenous spaces which are homotopy equivalent models for the complements, given in Table 2 and which we denote by K/L for each of the three cases. Also, the inclusions correspond to the following inclusions of spaces of matrices.

i) for the general case, $M_m(\mathbb{C}) \hookrightarrow M_{m+1}(\mathbb{C})$ by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$;

ii) for the symmetric case, $Sym_m(\mathbb{C}) \hookrightarrow Sym_{m+1}(\mathbb{C})$ sending $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$;

iii) for even $m = 2k$, $Sk_m(\mathbb{C}) \hookrightarrow Sk_{m+2}(\mathbb{C})$ sending $A \mapsto \begin{pmatrix} A & 0 \\ 0 & J_1 \end{pmatrix}$, for the 2×2 skew-symmetric matrix $J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Furthermore, for the cohomology of these spaces (via their homotopy equivalent compact models K/L for each case) the maps induced by the inclusions sends $e_j \mapsto e_j$ for the nonzero generators in successive spaces.

Via these inclusions, the corresponding actions of $GL_m(\mathbb{C})$ on these spaces (as explained in [D3]) applied to either I_m for the general or symmetric case, or J_k for the skew symmetric case factor through the homogeneous spaces given in Table 2 to give diffeomorphisms to the complements of $\mathcal{D}_m^{(*)}$ in each case. The inclusions of the homogeneous spaces correspond to the inclusions of the spaces of nonsingular matrices. Under this correspondence, the cohomology of the homogeneous spaces gives the cohomology of the complements of the spaces of $m \times m$ singular matrices $M_m^{(*)} \backslash \mathcal{D}_m^{(*)}$. Here we let $M_m^{(*)}$ denotes the space of $m \times m$ matrices of appropriate type.

Just as for Milnor fibers, we use multiplication to define a vanishing compact model. We let $\mathcal{P}^{(*)} \subset M_m^{(*)} \backslash \mathcal{D}_m^{(*)}$ denote the compact model for the complement in each of the three cases. The action of U_m in each case gives elements A of the compact model to be products of elements of U_m and hence $\|A\| = \sqrt{m}$. Thus, $\mathcal{P}^{(*)} \subset B_{\sqrt{m}}$. Then, we can multiply the spaces of matrices by nonzero constants a and for each case $a \cdot \mathcal{P}^{(*)} \subset B_{a\sqrt{m}}$. Then, for a neighborhood B_δ of 0 in $M_m^{(*)}$, if $a\sqrt(m) < \delta$, then $a \cdot \mathcal{P}^{(*)} \subset B_\delta \backslash \mathcal{D}_m^{(*)}$.

Then, we define $\Phi : \mathcal{P}^{(*)} \times (0, a) \to M_m^{(*)} \backslash \mathcal{D}_m^{(*)}$ sending $\Phi(A, t) = t \cdot A$. Then, Φ defines a vanishing compact model for the complement for each case.

It remains to show that the kite map of size ℓ detects the corresponding exterior algebra given in Table 3 for the characteristic cohomology of the complement. We consider $i_m^{(*)}(\ell) : \mathbf{K}_\ell(\mathbb{C}) \cap B_\varepsilon \to M_m^{(*)}$. If $M_\ell^{(*)}$ denotes the embedding of the corresponding $\ell \times \ell$ matrices given above, then there is an $a > 0$ so that $aM_\ell^{(*)} \subset \mathbf{K}_m(\ell) \cap B_\varepsilon$. Then, as in the proof of Theorem 7.1, the composition

$$a(M_\ell^{(*)} \backslash \mathcal{D}_\ell^{(*)}) \ \subset \ (\mathbf{K}_m(\ell) \backslash \mathcal{D}_m^{(*)}) \cap B_\varepsilon \ \xrightarrow{i_m^{(*)}(\ell)} \ M_m^{(*)} \backslash \mathcal{D}_m^{(*)}$$

induces in cohomology an isomorphism from the exterior subalgebra given in Table 3 to a subalgebra of the cohomology of $a(M_\ell^{(*)} \backslash \mathcal{D}_\ell^{(*)})$ (since it is diffeomorphic to $M_\ell^{(*)} \backslash \mathcal{D}_\ell^{(*)}$). As this homomorphism factors through $H^*(\mathbb{C}^n \backslash \mathcal{V}_0; \mathbf{k})$, it is also an isomorphism onto a subalgebra of this cohomology. This shows that $i_m^{(*)}(\ell)$ detects the exterior algebra, so by the second Detection lemma, the result follows for the complement.

Lastly, let $\widetilde{\Gamma}^{(*)}(f_0, \mathbf{k})$ denote the graded subspace of reduced homology obtained from the Kronecker dual $\Gamma^{(*)}(f_0, \mathbf{k})$ to this subalgebra. Then, by Alexander duality we obtain a graded subspace of $H^*(L(\mathcal{V}_0); \mathbf{k})$ isomorphic to $\widetilde{\Gamma}^{(*)}(f_0, \mathbf{k})$. It remains to show it is obtained from the exterior algebra by truncating it and applying an appropriate shift. As the exterior algebra satisfies Poincare duality under multiplication, this is done using the same argument in the proof of [D3, Prop. 1.9]. □

We reconsider the examples from §8

Example 9.3. In Example 8.1, we considered a singularity $\mathcal{V}_0$ defined by $f_0; \mathbb{C}^9, 0 \to Sym_4(\mathbb{C}), 0$ given by the matrix in Figure 4. It contains an unfurled kite map of size 3. We can apply Theorem 9.1.

Determinantal Hypersurface Type	$\mathcal{C}^{(*)}(f_0, \mathbf{k})$ contains subalgebra	Shift for Link
$\mathcal{D}_m^{sy}$ ℓ odd	$\Lambda^*\mathbf{k}\langle e_1, e_5, \dots, e_{2\ell-1}\rangle$	$2n - \binom{\ell+1}{2} - 2$
$\mathcal{D}_m^{sy}$ ℓ even	$\Lambda^*\mathbf{k}\langle e_1, e_5, \dots, e_{2\ell-3}\rangle$	$2n - \binom{\ell}{2} - 2$
$\mathcal{D}_m$	$\Lambda^*\mathbf{k}\langle e_1, e_3, \dots, e_{2\ell-1}\rangle$	$2n - \ell^2$ - 2
$\mathcal{D}_m^{sk}$ (m = 2k) ℓ even	$\Lambda^*\mathbf{k}\langle e_1, e_5, \dots, e_{2\ell-3}\rangle$	$2n - \binom{\ell}{2} - 2$

TABLE 3. The characteristic cohomology with coefficients in a field $\mathbf{k}$ of characteristic 0 for $\mathcal{V}_0 = f_0^{-1}(\mathcal{V})$ for each matrix type $\mathcal{V} = \mathcal{D}_m^{(*)}$. If f_0 contains an unfurled kite map of size ℓ, the characteristic cohomology $\mathcal{C}^{(*)}(f_0, \mathbf{k})$ contains an exterior subalgebra given in column 2 (where e_j has degree j). Then, for the link $L(\mathcal{V}_0)$, the characteristic cohomology contains as a graded subspace the exterior algebra in column 2 truncated in the top degree and shifted by the degree indicated in the last column. For the complements in the general or skew-symmetric cases, $\mathbf{k}$ in column 2 may be replaced by any coefficient ring R.

For coefficients a field $\mathbf{k}$ of characteristic 0, from Table 3 the characteristic cohomology of the complement of $\mathcal{C}^{(sy)}(f_0, \mathbf{k})$ contains an exterior algebra $\Lambda^*\mathbf{k}\langle e_1, e_5\rangle$, so there are $\mathbf{k}$-vector space generators e_1, e_5, and $e_1 \cdot e_5$ in degrees 1, 5 and 6.

The characteristic cohomology $\mathcal{B}^{(sy)}(f_0, \mathbf{k})$ of the link of $\mathcal{V}_0$, contains the subspace obtained by upper truncating the exterior algebra to obtain the $\mathbf{k}$ vector space $\mathbf{k}\langle 1, e_1, e_5\rangle$ and shifting by $2 \cdot 9 - 2 - \binom{4}{2} = 10$ to obtain 1-dimensional generators in degrees 10, 11, and 15. We note that the Link $L(\mathcal{V}_0)$ has real dimension 15, so a vector space generator of the characteristic subalgebra generates the top dimensional class.

We also note that from Table 2 that $\mathcal{D}_4^{(sy)}$ has link cohomology given by the upper truncated $\Lambda^*\mathbf{k}\langle e_1, e_5\rangle$ but shifted by $\binom{5}{2} + 4 - 2 = 12$ so there is 1 dimensional cohomology in degrees 12, 13, and 17. Thus, f_0^* does not send any of these classes to nonzero classes in the characteristic cohomology.

We do note that for the kite map $i_4^{(sy)}(3) : \mathbb{C}^7, 0 \to Sym_4(\mathbb{C}), 0$ the characteristic cohomology for the link is the upper truncated exterior algebra giving the $\mathbf{k}$ vector space $\mathbf{k}\langle 1, e_1, e_5\rangle$ and then shifted by 6. Thus, its degrees are 6, 7 and 11. We see that as noted in Remark 1.8, there is a shift in degrees given by twice the difference in dimension between each of the maps.

Second, we return to Example 8.2.

Example 9.4. From Example 8.2, the singularity $\mathcal{V}_0 = f_0^{-1}(\mathcal{D}_5)$ is defined by $f_0; \mathbb{C}^{21}, 0 \to M_5(\mathbb{C}), 0$, given by the matrix in Figure 5. Also, f_0 contains the linear kite map of size 4. Thus, we may apply Theorem 9.1, the characteristic cohomology $\mathcal{C}(f_0, R)$, for any coefficient ring R, contains the subalgebra $\Lambda^* R\langle e_1, e_3, e_5, e_7\rangle$. Hence, characteristic cohomology of the complement has R rank at least 1 in all degrees between 0 and 16, except for 2 and 14, and it is rank at least 2 in degree 8.

The characteristic cohomology $\mathcal{B}^{(sy)}(f_0, \mathbf{k})$ of the link contains the subspace obtained by upper truncating the exterior algebra over $\mathbf{k}$ obtained from the same $\mathbf{k}$ vector space by removing the generator of degree 16 given by the product $e_1{\cdot}e_5{\cdot}e_7{\cdot}e_9$. Then, we shift the resulting vector space by $2 \cdot 21 - 2 - 4^2 = 24$ to obtain 1-dimensional generators in all degrees between 24 and 39, except for 26 and 38, and it is dimension at least 2 in degree 32. We note that the Link $L(\mathcal{V}_0)$ has real dimension 39, so again a vector space generator of the characteristic subalgebra generates the top dimensional class.

10. Characteristic Cohomology for Non-square Matrix Singularities

We extend the results for $m \times m$ general matrices and matrix singularities to non-square matrices.

General $m \times p$ Matrix Singularities with $m \geq p$:
Let $M = M_{m,p}(\mathbb{C})$ denote the space of $m \times p$ complex matrices (where we will assume $m \neq p$, with neither $= 1$). We consider the case where $m > p$. The other case $m < p$ is equivalent by taking transposes. The varieties of singular $m \times p$ complex matrices, $\mathcal{D}_{m,p} \subset M_{m,p}(\mathbb{C})$, with $m \neq p$ were not considered earlier because they do not have Milnor fibers. However, the methods we applied earlier to $m \times m$ general matrices will also apply to the complement and link of $\mathcal{D}_{m,p}$. We explain that the complement has a compact homotopy model given by a Stiefel manifold. As for the case of $m \times m$ matrices, it has a Schubert decomposition using the ordered factorization by "pseudo-rotations" due to the combined work of J. H. C. Whitehead [W], C. E. Miller [Mi], and I. Yokota [Y]. The Schubert cycles give a basis for the homology and the Kronecker dual cohomology classes which can be identified with the classes computed algebraically in [MT, Thm. 3.10] (or see e.g. [D3, §8]). Thus, for appropriate coefficients, the form of both $\mathcal{C}_{\mathcal{V}}(f_0, R)$ and $\mathcal{B}_{\mathcal{V}}(f_0, \mathbf{k})$ can be given for $\mathcal{V} = \mathcal{D}_{m,p}$ and $f_0 : \mathbb{C}, 0 \to M_{m,p}(\mathbb{C}), 0$.

Then, we use the Schubert structure on the Stiefel manifolds to define vanishing compact models. This allows us to define, as for the $m \times m$ case, kite subspaces and maps to detect nonvanishing characteristic cohomology of the complement and link.

Complements of the Varieties of Singular $m \times p$ Matrices.
Let $M = M_{m,p}(\mathbb{C})$ denote the space of $m \times p$ complex matrices. The varieties of singular $m \times p$ complex matrices, $\mathcal{D}m, p$, with $m \neq p$ were not considered earlier because they do not have Milnor fibers. However, the methods do apply to the complement and link as a result of work of J. H. C. Whitehead [W]. We consider the case where $m > p$. The other case $m < p$ is equivalent by taking transposes. The complement to the variety $\mathcal{D}_{m,p}$ of singular matrices and can be described as the ordered set of p independent vectors in $\mathbb{C}^m$. Then, the Gram-Schmidt procedure replaces them by an orthonormal set of p vectors in $\mathbb{C}^m$. This is the Stiefel variety $V_p(\mathbb{C}^m)$ and the Gram-Schmidt procedure provides a strong deformation retract of the complement $M \backslash \mathcal{V}_{m,p}$ onto the Stiefel variety $V_p(\mathbb{C}^m)$. Thus, the Stiefel variety is a compact model for the complement.

Schubert Decomposition for the Stiefel Variety.
The work of Whitehead [W], combined with that of C. E. Miller [Mi], and I. Yokota [Y], provides a Schubert-type cell decomposition for $V_p(\mathbb{C}^m)$ similar to that given in the $m \times m$ case. There is an action of $GL_m(\mathbb{C}) \times GL_p(\mathbb{C})$ on $M_{m,p}(\mathbb{C})$ which

is appropriate for considering $\mathcal{K}_M$ equivalence of $m \times p$ complex matrix singularities. However, just for understanding the topology of the link and complement of $\mathcal{D}m, p$ it is sufficient to consider the left action of $GL_m(\mathbb{C})$ acting on M with an open orbit consisting of the matrices of rank p. As explained in [D4], the complement $M_{m.p}(\mathbb{C})\backslash\mathcal{D}_{m,p}$ is diffeomorphic to the homogeneous space $GL_m(\mathbb{C})/GL_{m-p}(\mathbb{C})$. The diffeomorphism is induced by $GL_m(\mathbb{C}) \to M_{m,p}(\mathbb{C})$ given by $A \mapsto A\cdot\begin{pmatrix} I_p \\ 0_{m-p,p}\end{pmatrix}$. Here the subgroup $GL_{m-p}(\mathbb{C})$ represents the subgroup of elements $\begin{pmatrix} I_p & 0 \\ 0 & A\end{pmatrix}$ with $A \in GL_{m-p}(\mathbb{C})$. This gives the isotropy subgroup of the left action on $\begin{pmatrix} I_p \\ 0_{m-p,p}\end{pmatrix}$.

For successive values of m, we have the successive inclusions: $GL_{m-1}(\mathbb{C}) \hookrightarrow GL_m(\mathbb{C})$ by $A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A\end{pmatrix}$. These induce inclusions $i_{m-1,p-1} : GL_{m-1}(\mathbb{C})/GL_{m-p}(\mathbb{C}) \hookrightarrow GL_m(\mathbb{C})GL_{m-p}(\mathbb{C})$. There is a corresponding inclusion of the spaces of matrices $M_{m-1,p-1}(\mathbb{C}) \hookrightarrow M_{m,p}(\mathbb{C})$ by $B \mapsto \begin{pmatrix} 1 & 0 \\ 0 & B\end{pmatrix}$. This inclusion induces a map of the complements of the varieties of singular matrices $\tilde{i}_{m-1,p-1} : M_{m-1,p-1}(\mathbb{C})\backslash\mathcal{D}_{m-1,p-1} \hookrightarrow M_{m,p}(\mathbb{C})\backslash\mathcal{D}_{m,p}$

The actions of the groups on the spaces of matrices commute via the inclusions of the groups with the corresponding inclusions of spaces of matrices. Thus, we have a commutative diagram of diffeomorphisms and inclusions

$$\begin{array}{ccc} GL_{m-1}(\mathbb{C})/GL_{m-p}(\mathbb{C}) & \xrightarrow{i_{m-1,p-1}} & GL_m(\mathbb{C})/GL_{m-p}(\mathbb{C}) \\ \simeq\downarrow & & \simeq\downarrow \\ M_{m-1,p-1}(\mathbb{C})\backslash\mathcal{D}_{m-1,p-1} & \xrightarrow{\tilde{i}_{m-1,p-1}} & M_{m,p}(\mathbb{C})\backslash\mathcal{D}_{m,p} \end{array} \tag{10.1}$$

The homogenenous spaces $GL_m(\mathbb{C})/GL_{m-p}(\mathbb{C})$ are homotopy equivalent to the homogenenous spaces given as the quotient of their maximal compact subgroups U_m/U_{m-p}. Via the vertical isomorphism in (10.1), it is diffeomorphic to the Stiefel variety $V_p(\mathbb{C}^m)$.

The Schubert cell decomposition of $V_p(\mathbb{C}^m)$ is given via ordered factorizations of matrices in U_m into products of "pseudo-rotations". For this we use the reverse flag with $\tilde{e}_j = e_{m+1-j}$ for $j = 1, \dots, m$ and $\mathbb{C}^k$ spanned by $\{\tilde{e}_1, \dots, \tilde{e}_k\}$. Then any $B \in U_m$ can be uniquely written by a factorization in decreasing order.

$$B \;=\; A_{(\theta_k,v_k)} \cdots A_{(\theta_2,v_2)} \cdot A_{(\theta_1,v_1)}\,, \tag{10.2}$$

with $v_j \in_{\min} \mathbb{C}^{m_j}$ and $1 \leq m_1 < m_2 < \cdots < m_k \leq m$, and each $\theta_i \not\equiv 0 \bmod 2\pi$. Here $v_j \in_{\min} \mathbb{C}^{m_j}$ means $v_j \in \mathbb{C}^{m_j}$ but $v_j \notin \mathbb{C}^{m_j-1}$. Also, each $A_{(\theta_j,v_j)}$ is a pseudo-rotation about $\mathbb{C} < v_j >$, which is the identity on $\mathbb{C} < v_j >^{\perp}$ and multiplies v_j by $e^{\theta_j i}$. In [D4, §3] the results are given for increasing factorizations,; however, as explained there, the results equally well hold for decreasing factorizations. If $m_{k'} > m - p \leq m_{k'+1}$, then each $A_{(\theta_j,v_j)}$ for $j > k$ belongs to U_{m-p}. Hence, B is in the same U_{m-p}-coset as

$$B' == A_{(\theta_k,v_k)} \cdots A_{(\theta_k,v_k)}\,.$$

Then, the projections $p_{m,p} : U_m \to U_m/U_{m-p}$ of the Schubert cells $S_{\mathbf{m}}$ for $\mathbf{m} = (m_1, \dots, m_k)$ with $m - p < m_1 < \cdots < m_k \leq m$ give a cell decomposition for

$U_m/U_{m-p} \simeq V_p(\mathbb{C}^M)$. Furthermore, the closures $\overline{S_{\mathbf{m}}}$ are "singular manifolds" the schubert cycles, which have Borel-Moore fundamental classes (see e.g. comment after [D4, Thm. 3.7]).

Cohomology of the Complement and Link.

We can give a relation between the homology classes given by the Schubert cycles resulting from the Whitehead decomposition and the cohomology with integer coefficients of the Stiefel variety, and hence the complement of the variety $\mathcal{D}_{m,n}$ (computed in [MT, Thm. 8.10a]).

Theorem 10.1. *The homology of the complement of $\mathcal{D}_{m,p}$ ($\simeq H_*(V_p(\mathbb{C}^m); \mathbb{Z})$) has for a free $\mathbb{Z}$-basis the fundamental classes of the Schubert cycles, given as images $p_{m,p\,*}(\overline{S_{\mathbf{m}}})$, with $\mathbf{m} = (m_1, m_2, \dots m_k)$ for $m - p < m_1 < \cdots m_k \leq m$, as we vary over the Schubert decomposition of U_m/U_{m-p}. The Kronecker duals of these classes give the $\mathbb{Z}$-basis for the cohomology, which is given as an algebra by*

$$H^*(M_{m,p} \backslash \mathcal{D}_{m,p}; \mathbb{Z}) \;\simeq\; \Lambda^*\mathbb{Z}\langle e_{2(m-p)+1}, e_{2(m-p)+3}, \dots, e_{2m-1}\rangle \tag{10.3}$$

with degree of e_j equal to j.

Moreover, the Kronecker duals of the simple Schubert classes $S_{(m_1)}$ *for $m - p < m_1 \leq m$ are homogeneous generators of the exterior algebra cohomology.*

Proof. The computation of $H^*(V_p(\mathbb{C}^m)$ is given in [D4, Thm. 3.7]. As it is a homotopy model for the complement (10.3) follows.

Second, that the Schubert cycles form a basis for the homology follows exactly as in the proof of [D4, Thm 6.1], as does the proof that the Kronecker duals to the simple Schubert cycles provide homogeneous generators of the exterior algebra. □

Cohomology of the Link.

As a consequence ofTheorem 10.1, we obtain the following conclusion for the link.

Theorem 10.2. *For the variety of singular $m \times p$ complex matrices, $\mathcal{D}_{m,p}$ (with $m > p$), the cohomology of the link is given (as a graded vector space) as the upper truncated cohomology $H^*(M_{m,p} \backslash \mathcal{V}_{m,p}, \mathbf{k})$ given in* (10.3) *and shifted by $p^2 - 2$.*

The Alexander duals of the Schubert cycles of nonmaximal dimension give a basis for the cohomology of the link.

Kite Spaces and Maps for $m \times p$ Matrix Singularities with $m \geq p$:

Definition 10.3. For $m \times p$ matrices with $m > p$, with $p \neq 1$ and the reverse standard flag of subspaces of $\mathbb{C}^m$, the corresponding *linear kite subspace of length* ℓ is the linear subspace of the space of matrices defined as follows: For $M_{m,p}(\mathbb{C})$, it is the linear subspace $\mathbf{K}_{m,p}(\ell)$ spanned by

$$\{E_{i,j} : r + 1 \leq i \leq m, r + 1 \leq j \leq p\} \cup \{E_{i,i} : 1 \leq i \leq r\}$$

where $r = p - \ell$.

Furthermore, we refer to the germ of the inclusion $i_{m,p}(\ell) : \mathbf{K}_{m,p}(\ell), 0 \to M_{m,p}(\mathbb{C}), 0$, for each of the three cases as a *linear kite map of size* ℓ. Furthermore, a germ which is $\mathcal{K}_M$ equivalent to $i_{m,p}(\ell)$ will be refered to as an *unfurled kite map of length* ℓ. We also say that a germ $f_0 : \mathbb{C}^n, 0 \to M_{m,p}(\mathbb{C}), 0$ contains a kite map of length ℓ if there is an embedding $g : \mathbf{K}_{m,p}(\ell), 0 \to \mathbb{C}^n, 0$ so that $f_0 \circ g$ is an unfurled kite map of colength ℓ.

$$\begin{pmatrix} * & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & * & 0 & \cdots & 0 \\ 0 & \cdots & 0 & * & \cdots & * \\ 0 & \cdots & 0 & * & \ddots & * \\ 0 & \cdots & 0 & * & \cdots & * \end{pmatrix}$$

FIGURE 6. Illustrating the form of elements of a linear kite space of length ℓ in the space of general $m \times p$ matrices with $r = p - \ell$. The upper $r \times r$ left matrix is a diagonal matrix with arbitrary entries, and the lower right matrix is a general matrix of size $(m - r) \times (p - r)$.

The general form of elements, "the kites" in the linear kite subspaces have the form given in (10.4) and the general element is exhibited in Figure 6.

Remark 10.4. Although the body of the kite is not square, the length ℓ denote the rank of a generic matrix in the body, which is consistent with the square case when $m = p$. We note that to be consistent with the form of the matrices for the group representation of the complement and the Schubert decomposition for the nonsquare case, the kite is "upside down". However, elements of $\mathcal{K}_M$ allow for the composition with invertible matrices GL_m and GL_p with entries in the local ring of germs. This allows for a linear change of coordinates so the kite can be inverted to the expected form as for the $m \times m$ case.

$$Q_{\ell, m-\ell} \;=\; \begin{pmatrix} D_r & 0_{m-r,p-r} \\ 0_{m-r,p} & A_{m-r,p-r} \end{pmatrix} \tag{10.4}$$

where $r = p - \ell$ and $A_{m-r,p-r}$ is an $(m - r) \times (p - r)$-matrix which denotes an arbitrary matrix in $M_{m-r,p-r}(\mathbb{C})$. Also, $0_{q,s}$ denotes a 0-matrix of size $q \times s$ and D_r, an arbitrary $r \times r$ diagonal matrix as in Figure 6. .

We have an analogue of the detection result for case of $m \times m$ general matrices.

Theorem 10.5. *Let $f_0 : \mathbb{C}^n, 0 \to M_{m,p}(\mathbb{C}), 0$ define a matrix singularity. If f_0 contains a kite map of length ℓ, then the characteristic cohomology of the complement $\mathcal{C}^{(*)}(f_0, \mathbf{k})$, for a field $\mathbf{k}$ of characteristic 0, contains the exterior algebra given by*

$$\Lambda^*\mathbf{k}\langle e_{2(m-p)+1}, e_{2(m-p)+3}, \ldots, e_{2(m-p)+2\ell-1}\rangle \tag{10.5}$$

and each e_j has degree j.

Furthermore, the characteristic cohomology of the link $\mathcal{B}^{(*)}(f_0, \mathbf{k})$, as a graded vector space contains the graded subspace given by truncating the top degree of the exterior subalgebra (10.5) of $\mathcal{C}^{(*)}(f_0, \mathbf{k})$ and shifting by $2n - 2 - \ell(2(m - p) + \ell)$.

For the complement $\mathbf{k}$ may be replaced by any coefficient ring R.

Proof. The line of proof follows that for the $m \times m$ general case.

Under the inclusion $i_{m-1,p-1} : V_{p-1}(\mathbb{C}^{m-1}) \hookrightarrow V_p(\mathbb{C}^m)$, the identification of the cohomology classes with Kronecker duals of the Schubert cycles implies

$$i^*_{m-1,p-1}(e_{2(m-p+j)-1}) = e_{2(m-p+j)-1} \text{ for } 1 \leq j \leq p-1 \text{ and } i^*_{m-1,p-1}(e_{2m-1}) = 0\,.$$

If we compose successive inclusions ℓ times to give $i_{m-\ell,p-\ell} : V_{p-\ell}(\mathbb{C}^{m-\ell}) \hookrightarrow V_p(\mathbb{C}^m)$, then the induced map on cohomology has image the algebra given in (10.5). Thus, given that $V_p(\mathbb{C}^m)$ provides a compact model for the complement, and the composition $i_{m-1,p-1} \circ i_{m-2,p-2} \circ \cdots \circ i_{m-r,p-r}$ with $r = p - \ell$ detects the subalgebra in (10.5).

Now using the vanishing compact model $t \cdot V_p(\mathbb{C}^m)$, we can follow the same reasoning as for the $m \times m$ case using the functoriality and invariance under $\mathcal{K}_{\mathcal{D}_{m,p}}$, we may apply the Second Detection Lemma to obtain the result.

Then, as the exterior algebra satisfies Poincare duality under multiplication, we can deduce the result for $\mathcal{B}_{\mathcal{D}_{m,p}}(f_0, \mathbf{k})$ using the same argument in the proof of [D3, Prop. 1.9] where for the shift $2n - 2 - \dim_{\mathbb{R}} K$ we replace $\dim_{\mathbb{R}} K$ by the top degree of the algebra (10.5). This is the same as $\dim_{\mathbb{R}} V_{p-r}(\mathbb{C}^{m-r})$, which is

$$2(p-r)((m-r)-(p-r)) + (p-r)^2 \;=\; (p-r)(2(m-p)-r) \;=\; \ell(2(m-p)+\ell)\,.$$

□

Example 10.6. Consider an example of a matrix singularity which is given by $f_0 : \mathbb{C}^{12}, 0 \to M_{5,4}(\mathbb{C}), 0$ defined by the matrix in Figure 7 for which all $g_{i,j}(\mathbf{x}, 0) = 0$. For $\mathbf{y} = (y_1, y_2, y_3, y_4)$, when $\mathbf{y} = 0$, we see that f_0 contains a kite map of

$$\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & g_{1,4}(\mathbf{x},\mathbf{y}) & g_{1,5}(\mathbf{x},\mathbf{y}) \\ x_{2,1} & x_{2,2} & x_{2,3} & g_{2,4}(\mathbf{x},\mathbf{y}) & g_{2,5}(\mathbf{x},\mathbf{y}) \\ g_{3,1}(\mathbf{x},\mathbf{y}) & g_{3,2}(\mathbf{x},\mathbf{y}) & y_3 & x_{3,4} & y_4 \\ y_1 & y_2 & g_{4,3}(\mathbf{x},\mathbf{y}) & g_{4,4}(\mathbf{x},\mathbf{y}) & x_{4,5} \end{pmatrix}$$

FIGURE 7. An example of a 5×4 matrix singularity f_0, with $g_{i,j}(\mathbf{x}, 0) \equiv 0$ for each (i,j). It contains a kite map of colength 2 given when all $y_i = 0$.

colength 2. Then, Theorem 10.5 implies that $\mathcal{C}_{\mathcal{D}_{4,3}}(f_0, \mathbb{Z})$ contains a subalgebra $\Lambda^*\mathbb{Z}\langle e_3, e_5\rangle$. Thus, also by Theorem 10.5, $\mathcal{B}_{\mathcal{D}_{5,4}}(f_0, \mathbb{Z})$ contains as a subgroup the subalgebra upper truncated and then shifted by $2 \cdot 12 - 2 - (4-2)(2(5-4)+2) = 14$. Thus, the classes $\{1, e_3, e_5\}$ are shifted by 14 to give classes in degrees $14, 17, 19$. As $\mathcal{V}_0 = f_0^{-1}(\mathcal{D}_{5,4}$ has codimension 2, the link $L(\mathcal{V}_0)$ has dimension 19 and the characteristic cohomology class in degree 19 generates the Kronecker dual to the fundamental class of $L(\mathcal{V}_0)$.

11. Cohomological Relations between Local Links via Restricted Kite Spaces

Lastly, it is still not well understood how the structure of the strata for the varieties of singular matrices contributes to the (co)homology of the links for the various types of matrices. We use kite spaces for all of the cases to determine the relation between the cohomology of local links for strata with the cohomology of the global link. This includes as well the relation between the local links for strata with local links of strata of higher codimension. This is via the relative Gysin homomorphism defined as eqrefEqn1.15 in §1, which is an analog of the Thom isomorphism theorem in these cases.

We do so by explaining how the kite subspaces provide transverse sections to the strata of the varieties of singular matrices for all three cases of $m \times m$ matrices and also for general $m \times p$ matrices. To consider all cases simultaneously, we denote the corresponding space of matrices by M and the variety of singular matrices by $\mathcal{D}_*^{(*)}$. Also, we consider the kite subspace of length ℓ of appropriate type which we denote by $\mathbf{K}_*^*(\ell)$. For the $m \times m$ cases, we also let $r = m - \ell$ (which is the same as $p - \ell$ when $m = p$) This will be the corank for generic matrices in the kite space.

We consider an affine subspace obtained by choosing fixed nonzero values at the entries in the tail (e.g. the value 1). When the entries in the body of the kite are 0, we obtain a matrix A of rank ℓ and hence corank r. Then, the resulting space has the form $A + M'$ where M' denotes one of the spaces $M_\ell(\mathbb{C})$, $M_\ell^{(sy)}(\mathbb{C})$, $M_\ell^{(sk)}(\mathbb{C})$, or $M_{m-r,p-r}(\mathbb{C})$ which is embedded, denoted by i, as the body of the kite. This provides a normal section to the stratum Σ_ℓ of matrices of corank ℓ through A. We refer to this affine subspace as a *restricted kit space*. We let $\mathcal{D}_*^{(*\,\prime)}$ denote the variety of singular matrices in M'. Then, in a sufficiently small tubular neighborhood T of Σ_ℓ we obtain $\mathcal{D}_*^{(*)} \cap T$ is diffeomorphic to $\Sigma_\ell \times (\mathcal{D}_*^{(*\,\prime)} \cap B_\varepsilon)$ for sufficiently small $\varepsilon > 0$. We refer to $L(\mathcal{D}_*^{(*\,\prime)})$ as the *local link of the stratum* Σ_ℓ.

Then i induces an inclusion $i : \mathcal{D}_*^{(*\,\prime)} \cap B_\varepsilon \subset \mathcal{D}_*^{(*)}$. There is the induced a map i^* on cohomology which sends the exterior algebra giving the cohomology of $M\backslash\mathcal{D}_*^{(*)}$ to the algebra (10.5). This is a consequence of the proofs of Theorems 9.1 and 10.5. Using this we have consistent monomial bases for the cohomology of the complement. This allows us to define consistent Kronecker pairings giving a well defined relative Gysin homomorphism. There is the following relation between the cohomology of the local link $L(\mathcal{D}_*^{(*\,\prime)})$ and the link $L(\mathcal{D}_*^{(*)})$.

Corollary 11.1. *The relative Gysin homomorphism*

$$i_* : H^*(L(\mathcal{D}_*^{(*\,\prime)}); \mathbf{k}) \to H^{*+q}(L(\mathcal{D}_*^{(*)}); \mathbf{k})$$

increases degree by $q = \dim_{\mathbb{R}} M - \dim_{\mathbb{R}} M'$*, which in the various cases equals for the* $m \times m$ *cases:* $2(m^2 - \ell^2)$ *for the general matrices;* $(m - \ell)(m + \ell + 1)$ *for symmetric matrices,* $(m - \ell)(m + \ell - 1)$ *for skew-symmetric matrices (with* m *and* ℓ *even); and for* $m \times p$ *matrices* $2(p^2 - \ell^2)$.

It is injective and sends the Alexander dual of the Kronecker dual of each monomial in the algebra (10.5) *to the corresponding Alexander dual the Kronecker dual of the same monomial but considered as an element of the cohomology of the complement* $M\backslash\mathcal{D}_*^{(*)}$.

Proof. By the above remarks, there is defined the relative Gysin homomorphism. If i denotes the inclusion of the reduced kite space into the space of matrices, then the induced map on cohomology of the complements, denoted i^*, (with coefficients $\mathbf{k}$ a field of characteristic 0) is surjective. We use the identification of the monomials $e_{\mathbf{m}}$ with the Kronecker duals denoted $e_{\mathbf{m}}^*$. Then, the inclusion i_* is the dual of i^*. Thus, the dual homomorphism for homology i_* is injective. When this is composed with Alexander isomorphisms (via the Kronecker pairings), it remains injective. By the properties of the corresponding cohomology classes of the links resulting from applying Alexander duality have the effect of raising degree by the difference $\dim_{\mathbb{R}} M - \dim_{\mathbb{R}} M'$ for each of the four types. These are then computed to give the stated degree shifts. $\square$

We also mention that there is an analogous version of this corollary for the case of the local link for a stratum $\Sigma_{\ell'}$ included in the local link of a stratum Σ_ℓ for $\ell' < \ell$. As an example we consider

Example 11.2. For the stratum $\Sigma_2 \subset Sym_5(\mathbb{C})$, the local link has reduced cohomology $\widetilde{\Lambda * \mathbf{k}}\langle e_1, e_5\rangle[4]$. However, the effect of Alexander duality on elements does not correspond to a shift. The reduced cohomology of the complement is spanned by the generators $\{e_1, e_1, e_1e_5\}$ with Kronecker duals denoted $\{e_{1\,*}, e_{5\,*}, (e_1e_5)_*\}$. Then, the corresponding to the Alexander dual generators denote $\{\tilde{e_1}, \tilde{e_5}, e_1\tilde{e}_5\}$ have degrees $9, 5, 4$ in that order. Also, the link $L(\mathcal{D}_5^{(sy)})$ has cohomology $\widetilde{\Lambda * \mathbf{k}}\langle e_1, e_5, e_9\rangle[13]$. Then, as i^* is surjective, i_* maps the Kronecker duals $\{e_{1\,*}, e_{5\,*}, (e_1e_5)_*\}$ to the corresponding elements for the complements $Sym_5(\mathbb{C})\ \mathcal{D}_5^{(sy)}$. Then these elements correspond to elements $\{\tilde{e'_1}, \tilde{e'_5}, (e_1\tilde{e}_5)'\}$ of degrees $27, 23, 22$. We see that the increase in degree is $2(15-6) = 18$.

Hence, one key point is to note that under the form of the cohomology represented as a subgroup by the truncated and shifted exterior algebras, the relative Gysin homomorphism does not map the shifted classes to the corresponding shifted classes.

Remark 11.3. For $m \times p$ finitely $\mathcal{K}_M$-determined matrix singularities $f_0 : \mathbb{C}^n, 0 \to M_{m,p}(\mathbb{C}), 0$, if $n \leq |2(m-p+2)|$, then by transversality, $\mathcal{V}_0$ has an isolated singularity and so has a Milnor fiber for any smoothing. As yet there does not appear to be a mechanism for showing how this Milnor fiber inherits topology from $M_{m,p}(\mathbb{C})$. However, for $(m,p) = (3,2)$, Frühbis-Krüger and Zach [F], [Z], [FZ] have shown that for the resulting Cohen-Macaulay 3-fold singularities in $\mathbb{C}^5$, the Milnor fiber has Betti number $b_2 = 1$, allowing the formula of Damon-Pike [DP3] to yield an algebraic formula for b_3. It remains to be understood how this extends to larger (m,p).

12. Module Structure for the Cohomology of Milnor Fibers of Matrix Singularities

We conclude by considering various contributions to the overall cohomology module structure of the Milnor fiber over the characteristic subalgebra.

We first consider two examples at the opposite extremes for matrix singularities Let $\mathcal{V}_0 = f_0^{-1}(\mathcal{V})$ be defined by $f_0 : \mathbb{C}^n, 0 \to M, 0$ for $\mathcal{V} = \mathcal{D}_m^{(*)}$. We illustrate how the characteristic subalgebra together with the topology of the "singular Milnor fiber" of f_0 contributes to the Milnor fiber cohomology, including the module structure, of $\mathcal{V}_0$.

Example 12.1. There two cases at opposite extremes for matrix singularities defined by $f_0 : \mathbb{C}^n, 0 \to M, 0$ which is transverse off 0 to $\mathcal{D}_m^{(*)}$. Then, either $n < \mathrm{codim}_M(\mathrm{sing}(\mathcal{V}))$ or f_0 is the germ of a submersion. In the first case, when $n < \mathrm{codim}_M(\mathrm{sing}(\mathcal{V}))$, $\mathcal{V}_0$ has an isolated singularity, and the singular Milnor fiber for f_0 is diffeomorphic to the Milnor fiber for $\mathcal{V}_0$, so the Milnor number of $\mathcal{V}$ and the singular Milnor number of f_0 agree. Also, $f_{0w}^*(e'_{\mathbf{m}}) = 0$ for all $e'_{\mathbf{m}}$ of positive degree; thus $\mathcal{A}^{(*)}(f_0, R) = H^0(\mathcal{V}_w; R) \simeq R$. As the Minor fiber is homotopy equivalent to a CW-complex of real dimension $n-1$, the corresponding classes which occur for the Milnor fiber will have a trivial module structure over $\mathcal{A}^{(*)}(f_0, R)$.

Second, if f_0 is the germ of a submersion, then the Milnor fiber has the form $F_w \times \mathbb{C}^k$, where F_w is the Milnor fiber of $\mathcal{V}$ for $k = n - \dim_{\mathbb{C}} M$. Thus, the Milnor fiber of $\mathcal{V}_0$ has the same cohomology as F_w. We conclude that $f_0^* : H^*(\mathcal{V}_w; R) \simeq H^*(F_w; R)$, or $\mathcal{A}^{(*)}(f_0, R) = H^*(F_w; R)$. Also, there are no singular vanishing cycles.

Thus, for these two cases there is the following expression for the cohomology of the Milnor fiber, where the second summand has trivial module structure shifted by degree $n-1$.

$$H^*(\mathcal{V}_w; R) \simeq \mathcal{A}^{(*)}(f_0, R) \oplus R^{\mu}[n-1] \tag{12.1}$$

where $\mu = \mu_{\mathcal{V}}(f_0)$ for $\mathcal{V} = \mathcal{D}_m^{(*)}$ denotes the singular Milnor number for the corresponding variety $\mathcal{V}_0$ of singular matrices.

Question: We ask how must (12.1) *be modified for matrix singularities of the three types?*

If R is a field of characteristic 0, then for a general hypersurface singularity we write (12.1) in a more general form as a direct sum.

$$H^*(\mathcal{V}_w; R) \simeq \mathcal{A}_{\mathcal{V}}(f_0, R) \oplus \mathcal{W}_{\mathcal{V}}(f_0, R) \tag{12.2}$$

We then may ask several questions about the properties of the summand $\mathcal{W}_{\mathcal{V}}(f_0, R)$.

i) Does $R^{\mu}[n-1]$ for $\mu = \mu_{\mathcal{V}}(f_0)$ occur as a summand?
ii) Does $\mathcal{W}_{\mathcal{V}}(f_0, R)$ vanish below degree $n-1$?
iii) If i) holds, is there an additional contribution in degree $n-1$ to $\mathcal{W}_{\mathcal{V}}(f_0, R)$?
iv) If ii) does not hold, can $\mathcal{W}_{\mathcal{V}}(f_0, R)$ be chosen to be an $\mathcal{A}_{\mathcal{V}}(f_0, R)$-submodule?

One step in establishing i) is in the case that f_0 has finite $\mathcal{K}_H$-codimension. By the H-holonomic property, f_0 is transverse to $\mathcal{V}$ in a neighborhood of $0 \in \mathbb{C}^n$. Then, there is a stabilization of f_0, $f_t : U \to M$ defined for $t \in (-\gamma, \gamma)$ for some $\gamma > 0$, so that for $0 < |t| < \gamma$ f_t is transverse to $\mathcal{V}$. Since $H \circ f_t$ defines a hypersurface, it satisfies the Thom condition a_f so for appropriate $0 < \eta << \delta$, we can stratify the mapping

$$H \circ f_t | \overline{B}_{\delta} : (H \circ f_t)^{-1}(B_{\eta}) \cap \overline{B}_{\varepsilon} \to B_{\eta} \,.$$

Then, the system of tubes for the stratification provide a neighborhood $N_{\mathcal{V}_t}$ of $\mathcal{V}_t = f_t^{-1}(\mathcal{V}) \cap B_{\varepsilon}$ and a retraction onto it (see e.g. [M1], [M2], or [GDW]). Given a Milnor fiber $\mathcal{V}_w = (H \circ f_t)^{-1}(w) \cap \overline{B}_{\varepsilon}$, let π denote the composition of the inclusion and the projection $\mathcal{V}_w \subset N_{\mathcal{V}_t} \to \mathcal{V}_t$. There is an induced homomorphism

$$\pi_* : H_*(\mathcal{V}_w; R) \;\to\; H_*(f_t^{-1}(\mathcal{V}) \cap \overline{B}_{\varepsilon}; R) \,. \tag{12.3}$$

In the case R is a field of characteristic zero as above, then if π_* is surjective, the dual map in cohomology (12.4) is injective.

$$\pi^* : H^*(f_t^{-1}(\mathcal{V}) \cap \overline{B}_{\varepsilon}; R) \;\to\; H^*(\mathcal{V}_w; R) \,. \tag{12.4}$$

Thus, by a result of Damon-Mond [DM], which also holds in the H–holonomic case [D1], $f_t^{-1}(\mathcal{V}) \cap \overline{B}_{\varepsilon}$ is homotopy equivalent to a bouquet of $\mu = \mu_{\mathcal{V}}$ spheres of dimension $n-1$. Thus, the injectivity of (12.4) gives the factor $\mathbf{k}^{\mu}[n-1]$ in (12.1).

This is just a first step in answering the above questions.

Partial Criterion for (12.2)*:* For the occurrence of $R^{\mu}[n-1]$ as a submodule of $\mathcal{W}_{\mathcal{V}}(f_0, R)$ in (12.2) for a finitely $\mathcal{K}_{HM}$-determined germ it is sufficient that (12.3) is surjective.

Singularity Type	**"Universal Singularity $\mathcal{V}$ "**	**Singularities of type $\mathcal{V}$**
Discriminants	Discriminants of Stable Germs	Discriminants of Finitely Determined Germs
Bifurcation Sets	Bifurcation Sets of $\mathcal{G}$-Versal Unfoldings	Bifurcation Sets of $\mathcal{G}$-Finitely Determined Unfoldings
Hyperplane Arrangements	Special Central Hyperplane Arrangements	Generic Versions of Special Hyperplane Arrangements
Hypersurface Arrangements	Special Central Hyperplane Arrangements	Hypersuface Arrangements of Special Type
Exceptional Orbit Hypersurfaces	Defined by Linear Algebraic Group Representations with Open Orbits	special types of determinantal arrangements:
Quiver Discriminants	Discriminants for Quiver Representations of Finite Type	Discriminants from Mappings to Quiver Representation Spaces
Cholesky-Type Factorizations	Discriminants for Cholesky-Type Factorizations	Discriminants for Cholesky-Type Factorizations for Matrix Families.
Matrix Singularities	Varieties of Singular $m \times m$ Matrices of Three Types	Matrix Singularities of Three Types

TABLE 4. Examples of General Cases of Singularities of Given Types.

For the remaining questions, there are few special cases such as generic central hyperplane arrangements [OR] and generic hypersurface arrangements [Li] where the answer to ii) is positive. However, there are significant additional contributions in degree $n-1$ to $\mathcal{W}_{\mathcal{V}}(f_0, R)$ (see §13).

Even for i) and $\mathcal{V} = \mathcal{D}_m^{(*)}$, this leaves the remaining issues to be addressed:

1) giving a sufficient condition that guarantees that (12.3) is surjective.
2) determining $\mu_{\mathcal{V}}$ for $\mathcal{V} = \mathcal{D}_m^{(*)}$. In the case that $\mathcal{V}_0$ has an isolated singularity (which requires that n is small, i.e. $n \leq \operatorname{codim}(\operatorname{sing}(\mathcal{D}_m^{(*)}))$, but allows arbitrary m), Goryunov-Mond [GM] give a formula in all three cases for $\mu_{\mathcal{V}}$ in terms of the formula of [DM] for free divisors with a correction term given by an Euler characteristic for a Tor sequence. Alternatively, by a different method using "free completions" in all three cases, with arbitrary n but for small m, Damon-Pike [DP3] give formulas for $\mu_{\mathcal{V}}$ as alternating sums of lengths of explicit determinantal modules. However, there stlll does not exist a formula valid for all m and n.
3) determining the form of additional module generators over $\mathcal{A}^{(*)}(f_0, R)$ besides those identified in (12.1).

13. Detecting Characteristic Cohomology for the General Case

Matrix singularities provide special examples of the general case of singularities of type $\mathcal{V}$, a hypersurface which represents a "universal singularity type". We summarize below the descriptions of several of the main classes of singularities of given universal singularity types in Table 4. These were mentioned in the introduction and all of them have characteristic cohomology for Milnor fibers, complements, and links. We can ask to what extent the form of the characteristic cohomology has been identified for each of these cases and when can the nonvanishing part be determined? We briefly comment on the cases and their relation with the results here.

Exceptional Orbit Hypersurfaces (yielding special determinantal arrangements): As listed in the tables, the form of the characteristic cohomology has been explicitly determined by the results in [DP] and [D3] for coefficients over a field of characteristics 0. This includes the cases of discriminants of quiver representation spaces of finite type and the discriminants for (modified) Cholesky-type factorizations. Singularities of these types are given by special types of "determinantal arrangements" given in [DP2]. For these cases compact models for Milnor fibers and complements are given as homogenenous spaces and can be used to define vanishing compact models in [DP]. Then, the tower structures given in [DP2] can be used to give analogous versions of kite maps for these cases which can be used for detection criteria.

For the representation spaces for quiver of finite type given in [BM], there are also compact models as given in [D3] which can be used to construct vanishing compact models. Now the restrictions of the root structures in [BM] need to be employed to define detection maps.

In both cases the details still have to be determined for identifying nonvanishing parts of the characteristic cohomology.

Central Hyperplane and Hypersurface Arrangements :

For a central hyperplane arrangement $\mathcal{V} \subset \mathbb{C}^N$, it follows from the work of Arnold [A], Brieskorn [Br], and Orlik-Solomon [OS] (more generally [OT, Chaps. 3, 5]), there is an explicit description of the cohomology of the complement $H^*(\mathbb{C}^N \backslash \mathcal{V}; \mathbb{C})$ generated by 1-forms corresponding to each hyperplane with combinatorially defined relations (in fact, by Brieskorn, this holds for coefficients $\mathbb{Z}$ using the $\mathbb{Z}$-subalgebra on these generators). For a central hyperplane arrangement $\mathcal{V}_0 \subset \mathbb{C}^n$ defined by a linear map $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^N, 0$ transverse to $\mathcal{V}$ off $0 \in \mathbb{C}^n$, it then follows from transversality that the combinatorial conditions up to codimension $n-1$ continue to hold. It follows that $H^*(\mathbb{C}^n \backslash \mathcal{V}_0; \mathbb{C}) = \mathcal{C}_{\mathcal{V}}(f_0, \mathbb{C})$. This then allows us to compute $\mathcal{B}_{\mathcal{V}}(f_0, \mathbb{C})$ by adding relations in degree $n-1$ and above; and then $H^*(L(\mathcal{V}_0); \mathbb{C}) = \mathcal{B}_{\mathcal{V}}(f_0, \mathbb{C})$ can be explicitly computed.

In the case that f_0 is nonlinear there is no general result for $\mathcal{C}_{\mathcal{V}}(f_0, \mathbb{Z})$, although we know the form it has as the image $f_0^*(H^*(B_\delta^N \backslash \mathcal{V}; \mathbb{Z}))$ for sufficiently small $\delta > 0$. The problem for determining this image involves detecting the nonvanishing of the terms. One result is obtained by Libgober [Li] for the case where $\mathcal{V} = \mathcal{B}_N$, the Boolean arrangement. The singularities $\mathcal{V}_0$ are referred to by him as *isolated non-normal crossings* (INNC) (these are the same as hypersurface arrangements defined by a finitely $\mathcal{K}_{\mathcal{B}_N}$-determined germ f_0 [D1]). Then, $\mathbb{C}^N \backslash \mathcal{B}_N$ is homotopy equivalent to a torus T^N so

$$H^*(\mathbb{C}^N \backslash \mathcal{B}_N; \mathbb{Z}) \simeq \Lambda^* \mathbb{Z} < e_1, \dots, e_N > .$$

The result of Libgober [Li, Thm 2.2] gives results for the homotopy groups, which together with the relative Hurewicz Theorem and the universal coefficient theorem, implies that $\mathcal{C}_{\mathcal{V}}(f_0, \mathbb{Z})$ contains $\Lambda^*\mathbb{Z} < e_1, \dots, e_N >$ up through degrees $\leq n-2$, with no information on dimension $n-1$.

However, there does not exist a general result guaranteeing the nonvanishing of the characteristic cohomology for generic hypersurface arrangements based on a general central hyperplane arrangement. In the case of complexified arrangements, the Salvetti complex, see e.g. [OT, §5.2], provides a compact model for the complement, which then provides a vanishing compact model for the detection method. Hence, detection maps can be defined by linear sections whose images contain appropriate subspaces of the Salvetti complex. If $df_0(0) : \mathbb{C}^n \to \mathbb{C}^N$ contains a generic k-plane section, then f_0 plays the role of a kite map; and the detection method will imply that $H^*(\mathbb{C}^N \backslash \mathcal{B}_N; \mathbb{Z})$ will map isomorphically in degree $< k-1$ onto its image in the characteristic cohomology of the complement.

Milnor Fibers of Hyperplane and Hypersurface Arrangements: For the cohomology of the Milnor fiber of central hyperplane arrangements, there are basically very few results. For central generic arrangements, the cohomology has been determined by Orlik-Randell. The Milnor fiber of B_N has the homotopy type of a torus of dimension $N-1$ so its cohomology has the form $\Lambda^*\mathbb{Z} < e_1, \dots, e_{N-1} >$. Orlik Randell [OR, Thm 2.6] show that this maps isomorphically to $H^*(\mathcal{V}_w; \mathbb{Z})$ in degrees $< n-1$ and in degree $n-1$ the Betti number is $b_{n-1} = \binom{N-2}{n-2} + N\binom{N-2}{n-1}$. It follows the characteristic cohomology contains all of $\Lambda^*\mathbb{Z} < e_1, \dots, e_{N-1} >$ up through degree $n-2$.

There is an analogous result for generic hypersurface arrangements, i.e. INNC, by a result of Libgober [Li, Prop. 4.6] which also implies that the characteristic cohomology of the Milnor fiber contains all of $\Lambda^*\mathbb{Z} < e_1, \dots, e_{N-1} >$ up through degree $n-2$. However, he does not give an explicit formula for b_{n-1}. Both of these results use a covering representation of the Milnor fiber to carry out the computations. This was extended by Cohen-Suciu [CS] to more general hyperplane arrangements; however, their computation involves complexes of chains for local systems on the covering representation. This allows them to compute explicitly the result for certain hyperplane arrangements in dimension ≤ 3, but there are not general results.

These show that for the generic linear arrangements and hypersurface arrangements the characteristic cohomology for the Milnor fiber occupies all degrees below $n-1$, so for these cases the answer to question ii) (in §12) is positive. We also ask for the extent of the additional contribution to $\mathcal{W}_{\mathcal{V}}(f_0, R)$ in (12.2). As B_N is a linear free divisor, we can compute $\mu_{B_N}(f_0)$ using the calculations in [D1, §6]. For the generic hyperplane arrangement case, we have $\mu_{B_N}(f_0) = \binom{N-1}{n}$. Also, in degree $n-1$, the characteristic cohomology can contribute a subspace of dimension $\binom{N-1}{n-1}$. Then, b_{n-1} can be reexpressed in terms of these two dimensions by: $b_{n-1} = \binom{N-1}{n-1} + n\binom{N-1}{n}$. It follows that if the characteristic cohomology contributes the full amount in degree n-1, then there is still an additional contribution to $\mathcal{W}_{\mathcal{V}}(f_0, R)$, beyond tthat from the singular Milnor fiber, of dimension $(n-1)\binom{N-1}{n}$. This says that each singular vanishing cycle contributes n vanishing cycles to the Milnor fiber. This raises the question of how exactly this extra cohomology is realized geometrically.

For a generic hypersurface arrangement $\mathcal{V}_0, 0$ defined by a nonlinear map germ f_0 transverse to $B_N, 0$ off $0 \in \mathbb{C}^n$, there are less precise results, even though we know the form of $\mathcal{C}_{\mathcal{V}}(f_0, \mathbb{C})$ and $\mathcal{B}_{\mathcal{V}}(f_0, \mathbb{C})$ by the above. To detect nonvanishing contributions to the characteristic cohomology for the Milnor fiber using the method given here, requires vanishing compact models for the Milnor fiber which we do not have.

We are able to give one type of example where we can explicitly see what occurs in dimension $n-1$.

Example 13.1. We consider an isolated curve singularity $\mathcal{V}_0, 0 \subset \mathbb{C}^2, 0$ defined by $f = f_1 \cdot f_2 \cdots f_k$ with each f_j defining an isolated curve singularity $\mathcal{V}_i$, so $\mathcal{V}_0 = \cup_{i=1}^k \mathcal{V}_i$. We can alternately consider $\mathcal{V}_0$ as a generic hypersurface arrangement defined by $F = (f_1, \dots, f_k) : \mathbb{C}^2, 0 \to \mathbb{C}^k, 0$ for the Boolean arrangement $\mathcal{B}_k \subset \mathbb{C}^k$. We note that $\mathbb{C}^2$ lies below the dimension to which the result of Libgober applies.

We can stabilize F to $F_t = (f_{1\,t}, \dots, f_{k\,t}) : U \to \mathbb{C}^k$ so in particular each $\mathcal{V}_{j\,t} = f_{j\,t}^{-1}(0) \cap B_\varepsilon$ is a Milnor fiber for f_j and the $\mathcal{V}_{j\,t}$ pairwise intersect transversely. Then, $\mathcal{V}_{0\,t} = \cup_{i=1}^k \mathcal{V}_{j\,t}$ is the singular Milnor fiber for F. It is homotopy equivalent to a bouquet of $\mu_{\mathcal{B}_k}(F)$ S^1's. If $I(f_i, f_j)$ denotes the intersection number of $\mathcal{V}_{i\,t}$ and $\mathcal{V}_{j\,t}$. A smooth nearby fiber of f close to $\mathcal{V}_{0\,t}$ adds one vanishing cycle for each intersection point. Thus, the Milnor number of f is given by

$$\mu(f) \;=\; \mu_{\mathcal{B}_k}(F) \,+\, \sum_{i<j} I(f_i, f_j)\,. \tag{13.1}$$

Then, $\mathcal{B}_k$ has a Milnor fiber which is homotopy equivalent to a $k-1$ torus and has the torus as a compact model. Thus, the possible contribution to $\mathcal{A}_{\mathcal{B}_k}(F, \mathbb{Z})$ in dimension 1 would have rank $k-1$. However, for most examples the sum of intersection numbers considerably exceeds $k-1$; thus, $\mathcal{W}_{\mathcal{B}_k}(F, \mathbb{Z})$ must be considerably larger than the contribution from characteristic cohomology. For example if $f = f_1 \cdot f_2 \cdot f_3$ with the f_i distinct generic quadrics, then $\mu(f) = 25$, by [D1, §6] $\mu_{\mathcal{B}_3}(F) = 13$, and the sum of intersection numbers is 12; while $3 - 1 = 2$. Thus, most of the cohomology in dimension 1 not part of the singular Milnor fiber lies outside of the characteristic cohomology.

A basic question then is to determine geometrically what part of the characteristic cohomology exists in $\mathcal{W}_{\mathcal{B}_k}(F, \mathbb{Z})$ and what geometrically accounts for the remainder

Discriminants and Bifurcation Sets :

There are only very limited results for the topological structure of the complement for either types. For the stable germs obtained by unfolding simple hypersurface singularities, the complement is a $K(\pi, 1)$ by results of Arnold and Brieskorn. However, this does not continue to be always true for ICIS by Knörrer. Also, there is an explicit basis for the cohomology of complements of discriminants of stable A_k-singularities by results of Fuks and for those of types D, and for types B and C for functions on manifolds with boundaries, by Goryunov. Hence, only for complements of discriminants of finitely determined germs of these types do we have the form for $\mathcal{C}_{\mathcal{V}}(f_0, \mathbb{C})$. Otherwise little is known about the characteristic cohomology for these singularities.

Also, there are many different equivalence groups $\mathcal{G}$ in the holomorphic category , allowing additional features to be preserved such as (flags of) distinguished parameters, equivariant germs, diagrams of mappings, distinguished varieties, and restrictions to (flags of) subvarieties, etc. These are geometric subgroups of $\mathcal{A}$ or $\mathcal{K}$. Then, unfoldings of finitely $\mathcal{G}$-determined germs are modeled on the $\mathcal{G}$-versal unfoldings. These need not always be hypersurfaces; however, in many important cases they are. For virtually all of these, the cohomology of the Milnor fiber (in the hypersurface case) and that of the complement and link is unknown. Hence even the form of the characteristic cohomology is unknown. Because of such a great variety of possibilities, essentially nothing is known about the topology of bifurcation sets of unfoldings for any of these groups $\mathcal{G}$.

By contrast, many of the universal singularities have been shown to be (H-holonomic) free divisors, see e.g. the list in [D1] and the additional work in e.g. [GM] and [DP2]. Thus, for these we can compute the singular Milnor number to determine a possible contribution for the Milnor fiber using the results of the previous section.

Hence, many of the list of questions given for matrix singularities still remain to be resolved in the other cases mentioned about.

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